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The generation of mass in a non-linear field theory

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Abstract: The mass spectrum of elementary particles is calculated in a new approach, based on B. Heim's quantum field theory, which manifests in a non-linear eigenvalue equation and merges into the Einstein field equation in the macroscopic limit. The poly-metric of the theory allows spacetime and matter to be described in a unified formalism, representing a radical geometrisation of physics. The calculated mass energies are in very good agreement with the empirical data (error <1% on average) if the mass scale is gauged to the electron as lowest mass and the second main parameter, determining the strength of obtained mass hierarchy levels, is close to the half inverse of the fine structure constant, describing the difference in strength between the electromagnetic and the strong interaction. The obtained hierarchy levels are not identical to the particle generations of the Standard Model; however, show a self-similarity typical for non-linear theories. For higher values of the main quantum number N, the calculated mass formula becomes identical to the phenomenological formulae of Nambu, respectively, Mac Gregor.

Keywords: general relativity; geometrisation; mass spectrum of elementary particles; non-linear field theory.

1 Introduction and phenomenology

The Standard Model (SM) of elementary particles, consisting of the U(1)/SU(2) spontaneously symmetry-broken unified electro-weak interaction and the SU(3) Quantum Chromodynamics (QCD), is currently our most successful fundamental theory, explaining a wide range of phenomena of particles and in high-energy physics [1]. However, it leaves important questions open which it cannot explain: Why are there three generations of elementary particles (leptons, quarks)? What about the masses of these particles? The SM cannot derive them. The Higgs mechanism provides a model of breaking the electro-weak symmetry and attaching mass to particles, but it leaves the number of free parameters the same, as there is a Yukawa coupling constant for each mass to be determined.[1] Why does the mass spectrum consist of a light sector with the neutrinos and the electron, of a middle sector with the μ- and τ-lepton and the hadrons (mesons and baryons) and finally of the top quark and the heavy bosons? Where does mass come from at all?[2]

As stated, the Higgs mechanism alone does not give a satisfactory answer. To quote Richard Feynman (as in [2]): "Throughout this entire story there remains one especially unsatisfactory feature: the observed masses of the particles, m. There is no theory that adequately explains these numbers. We use the numbers in all our theories, but we do not understand them – what they are, or where they come from. I believe that from a fundamental point of view, this is a very interesting and serious problem." [4], p. 152. The well-known fundamental theory candidates as GUTs, Supersymmetry, Superstring and M-theory are not (yet) able to give answers on the issues of the generations and masses and possibly will never do it ([2, 5]). Finally, the SM is a theory on a fixed Lorentz invariant 4-dimensional spacetime, leaving gravitation and general covariance completely out.

In recent years considerations have been made ([2, 6–9]) to find paths to alternative approaches which would go beyond the SM and provide new insight in the raised questions. On the other hand, there are empirical or phenomenological observations about the mass spectrum which have not played a big role on the main path of research in the past, but which are too significant to be ignored and which could give a hint into the direction of a valuable approach ([7, 10–27] incl. references):

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1 We quote Hansson [2]: "The Higgs mechanism simply replaces one ad hoc mass parameter (m_i) with another equally ad hoc Yukawa coupling constant (λ_i) to the Higgs field, and because of this we are free to make other hypotheses as to what physically generates mass. A Higgs-like symmetry breaking mechanism is the favourite way to break also other (hypothetical) symmetries, such as supersymmetry. To us that seem to be the wrong way to proceed, as such a symmetry-breaking mechanism always introduces new free (ad hoc) parameters, and a more fundamental theory should contain fewer free parameters, not more."

2 See [3] for an in-depth discussion on the concepts of mass.

As summarised by Varlamov in [7], already in 1952 Nambu [10] gave attention to the existence of empirical "Balmer-like" relations in the mass spectrum of elementary particles:

$$m_N = \frac{N}{2}\, 137\, m_{\mathrm{e}} \;\approx\; N \cdot 35\,\mathrm{MeV} \tag{1}$$

with N being a positive integer number and m_{e} the rest mass of the electron. Further, in 1979, Barut [11] proposed a mass formula for the leptons:

$$m_N = m_{\mathrm{e}} \left(1 + \frac{3}{2}\alpha^{-1} \sum_{n=0}^{N} n^4 \right) \tag{2}$$

where $\alpha \approx 1/137$ is the fine structure constant. According to (2), the masses of the electron, muon and τ-lepton are defined at $N = 0$, 1 and 2, respectively. Later on, empirical relations of the form (1) were studied by many authors (see [12–27] and further citations in [7]). The Nambu formula (1) can be written also via the fine structure constant: $m = \frac{N}{2\alpha} m_{\mathrm{e}}$.

This formula leads to a so-called alpha-quantisation of the elementary particle masses, see [17, 27]. Greulich [27] emphasised that the fact that such a very simple equation, solely based on α, predicts particle masses with high accuracy (approx. 1% for masses of 11 fundamental elementary particles he considered, those with a lifetime $> 10^{-24}$ s), indicates that this formula has a real physical background.

It should be complemented that Mac Gregor observed an alpha-dependent mass relation also between the light quarks and the gauge bosons and the top quark [18]: He summarises that the building blocks in the three α-quantized particle production channels are

$$\begin{aligned} &\text{boson } (J = 0)\text{: } m_{\mathrm{e}}/\alpha = m_{\mathrm{b}} = 70.025\,\mathrm{MeV} \\ &\text{fermion } (J = 1/2)\text{: } 3m_{\mathrm{e}}/2\alpha = m_{\mathrm{f}} = 105.038\,\mathrm{MeV} \\ &\text{gauge boson } (J = 1/2)\text{: } m_{\mathrm{u,d}}/\alpha = m_{\mathrm{gb}} = 43.17\,\mathrm{GeV}, \end{aligned} \tag{3}$$

and that the idea of a linkage between the low-energy mass spectrum and the very-high-energy gauge bosons and the top quark is also suggested by the unexpected experimental discovery of a mass relationship between the gauge bosons and top quark, $m_{W^\pm} + m_{Z^0} = m_t$ (1.1% accuracy).

Two further models shall be mentioned which also provided compact mass formulae: In 2003, Sidharth [28, 29] proposed the following half-empirical formula which he derived from a QCD-based potential approach, leading to a harmonic oscillator-like result with the (positive integer) quantum numbers m and n:

$$m_{\mathrm{P}} = m \left(n + \frac{1}{2} \right) m_\pi \tag{4}$$

The Sidharth formula (4) describes the mass spectrum of mesons and baryons (known up to 2003) with an accuracy of 3% [29]. Varlamov derived a mass spectrum of localized states (elementary particles) of a single quantum system where these states are understood as cyclic representations of a fundamental Lorentz group symmetry. He obtained masses of the leptonic (except neutrinos) and hadronic sectors proportional to the rest mass of the electron

$$m_{\mathrm{P}}^{(s)} = \left(l + \frac{1}{2} \right) \left(l' + \frac{1}{2} \right) m_{\mathrm{e}} \tag{5}$$

with an average accuracy of 0.41% [7]. The l, l' are integer or half-integer numbers, the spin of each state is given by $s = |l - l'|$.

So, these observations and models cover the leptonic, hadronic and gauge boson sector. For the hadronic sector, where the quark picture and QCD are the accepted theories within the SM, we quote from [7] "it is well known that a quark model, based on the flavour SU(3) group, does not explain the mass spectrum of elementary particles ... In nature we see a wide variety of baryon octets (see, for example, Particle Data Group: pdg.lbl.gov) where mass distances between these octets are not explained by the given SU(3)- and SU(6)-mass formulae. As a rule, all the predicted masses in SU(3)- and SU(6)-theories have a low accuracy (on an average 4–6%)". As summarised in [27], better accuracies are only achieved when a theory is restricted to subsets of the elementary particles, for example the octet of mesons (the two pions and the four kaons) or the decuplet of baryons $(\Delta, \Sigma, \Xi, \Omega)$. Very good mass predictions have been obtained for the proton and its excited states. As the best, the AdS/QCD model using di-quarks has an accuracy of 2.5% [30]. See [1] for the current status of quark model calculations and references, i.e. [31–34], and also [35] for an overview of the theories (in German language).

Being much more calculations from first principles of QCD, lattice calculations, with uncertainties due to the finite lattice of meanwhile less than 1%, meet the empirical masses of 4% or better ([36, 37]). But they typically depend on 3–4 parameters which fix the light quark masses (u, d and s) and the scaling (e.g. in 2 + 1 flavour calculations by fitting the π, the K mass and one of the heavier baryons Ω or Ξ [36]). In a 2013 calculation by Dudek et al. [38] on the excited isoscalar meson spectrum the quark mass is still heavier than its physical value and the pion mass too high (392 MeV).

2 Hansson's perturbative model combined with the phenomenology

Hansson [2] proposes that the mass of a particle has a strictly local origin, arising from its self-interaction(s): "That is, the mass is equivalent to the energy contained in the associated gauge fields (in perturbative quantum field theory; the energy of the "cloud" of virtual gauge particles). Such a connection between fundamental dynamical interactions and mass seems only reasonable as only mass is needed to go from kinematics to dynamics [3]. As most of the (at least) 18 arbitrary parameters of the "standard model" arise because of the mass problem, a connection between the masses and the ordinary, non-Yukawa, i.e. non-Higgs interaction couplings would also significantly reduce the number of free ad hoc parameters", he argues.[3] With further arguments given in [2], taking into account that perturbatively formally infinite quantities like mass are in nature not singular and that it obviously can be concluded that the stronger the self-interaction, the more massive the particle will be, Hansson suggests the ansatz

$$m = B_i\left(Q^2\alpha + T^2\alpha_{\text{weak}} + C^2\alpha_{\text{s}}\right) := m_{\text{em}} + m_\nu + m_{\text{s}} \quad (6)$$

for the mass of a particle (to order $O(\alpha)$). α is the fine structure constant and α_{weak} and α_{s} are the (low-energy) couplings of the weak and strong interactions. Q gives the particle's electrical charge in units of e, and T and C are analogous quantities (of order one) coming from the gauge-groups for weak charge and colour charge. B_i is a normalising constant, which in a truly non-perturbative treatment of the standard model should be calculable. The Index i refers to the 3 different generations of particles. Hansson emphasises that a physical mechanism for connecting the different B_i would be highly desirable and that, still, the number of arbitrary parameters has decreased compared to the Higgs-mechanism due to the relation between coupling strengths and masses, which in the standard model are completely independent quantities.

3 And Varlamov reminds: 13 of these constants are directly related to the fermion masses, namely, 3 lepton masses, 6 quark masses and 4 mixing angles. All these mass parameters have to be adjusted according to experimental measurements and cannot be predicted within the theory (SM) [7].

Formula (6) gives calculated relations between particle masses $m_\nu/m_{\text{e}} \sim 10^{-7}$, $m_{\text{q}}/m_{\text{e}} \sim 10^2$, or in absolute mass units $m_\nu \sim 0.1$ eV, $m_{\text{q}} \sim 10$ MeV, compatible with experiments, respectively, current quark masses [2].

If we now combine this ansatz with the phenomenology described in the previous section, we can derive the following relations for the hadrons and leptons, leaving out the neutrinos in this consideration,

$$\frac{m_{\text{s}}}{m_{\text{em}}} = \frac{\alpha_{\text{s}}}{\alpha}\frac{C^2}{Q^2} := \frac{N}{\alpha}, \qquad \Rightarrow m_{\text{s}} = N\frac{m_{\text{em}}}{\alpha} \quad (7)$$

for each generation, when taking into account that $\alpha_s \sim 1$ and N being an integer or rational number. For $m_{\text{em}} = m_{\text{e}}$ we get $m_{\text{e}}/\alpha \sim 70$ MeV, as appearing in formulae (1)–(3) above.

Between the 3 electromagnetic leptons we find the empirical relations $m_\mu =\sim 105\,\text{MeV} \sim \frac{3}{2}70\,\text{MeV} = \frac{3}{2}\frac{m_{\text{e}}}{\alpha}$, applied to (6) this gives $\frac{m_\mu}{m_{\text{e}}} = \frac{B_2(\mu)}{B_1(e)} = \frac{3}{2}\frac{1}{\alpha}$ and with $B_1(e) = \frac{m_{\text{e}}}{\alpha}$ from (6) it follows

$$B_2(\mu) = \frac{3}{2}\frac{m_{\text{e}}}{\alpha^2}, \quad (8)$$

and from $m_\tau \sim \frac{3}{2}17\frac{m_{\text{e}}}{\alpha}$ (empirically)

$$B_3(\tau) = \frac{3}{2}17\frac{m_{\text{e}}}{\alpha^2}. \quad (9)$$

Combining Eqs. (6)–(9) and inserting the charges (respectively integrating the colour charges in the N_{is}) we derive the following phenomenological mass formula (without neutrinos) per generation i:

$$m_i = \frac{m_{\text{e}}}{2}\left(\frac{N_{i\text{l}}}{\alpha^{n_{i\text{l}}-1}} + \frac{N_{i\text{s}}}{\alpha^{n_{i\text{s}}}}\right) \quad (10)$$

with $N_{1\text{l}} = 2, n_{1\text{l}} = 1, N_{2\text{l}} = 3, n_{2\text{l}} = 2, N_{3\text{l}} = 3\cdot 17, n_{3\text{l}} = 2$ for the leptons (e, μ, τ), $n_{1\text{s}} = 1, n_{2\text{s}} = 1, n_{3\text{s}} = 2$ for the strong interacting particles, and for a single mass term

$$m_j = \frac{m_{\text{e}}}{2}N_j(1/\alpha)^{n_j}, \qquad n_j = 0, 1, 2. \quad (11)$$

The $N_{i\text{s}}$ are still to be determined, depending on the respective hadron particles (=0 for all leptons). The $n_{2\text{s}}$ and $n_{3\text{s}}$ are set according to the empirical situation, where $n_{2\text{s}} = 1$ does not follow the power of $1/\alpha$ expected from the analogue of $B_2(\mu)$, but $n_{3\text{s}}$ does (see $B_3(\tau)$). $n_{3\text{s}} = 2$ is obviously relevant at least for the top quark level and seems to fit also for the level of the massive gauge bosons, compare [18] and see Tables 2 and 3 in Appendix N. The reason for this is an open question in our current picture, as the gauge bosons do not underlie the strong interaction (α_{s}).

Nonetheless, with (10) we have achieved a result which already reduces the number of parameters, meets the empirical data within an appropriate range (see calculation E_4 in Tables 1 and 2) and includes the insight that the relation of α to the other coupling constants α_{weak} and α_{s} seems to transform into the relations of the particle masses.

However, the goal should be to find a candidate for a fundamental theory which can deliver a respective formula derived and calculated from its first principles. Again referring to Hansson's thoughts [2], this theory should be non-linear, as in the SM even in the abelian QED the evolution equations constitute a non-linearly coupled system (see Eqs. (10)–(13) in [2]), although in quantum field theory the used dominating and inherently perturbative Feynman diagram techniques mask the non-linearities and produce a behaviour that closely mimics the linear case.

We shall later see (in Section 4) that apparently only a non-linear theory can "create" a hierarchic mass spectrum, so we follow Hansson's assessment that this will be an essential feature of a "true" theory.[4] One typical feature of non-linear equations is the multiplicity of solutions, which may let us hope that different generations of particles come out automatically ([2]).

Beside this, the theory of course needs to possess the features required from a theory capable in describing the known phenomena in the domain of the SM, especially regarding quantum theory, the electromagnetic interaction and relativity (it should lead to Dirac's and Maxwell's equations in the respective limits). It would undoubtedly be considered as an extraordinary advancement if such a theory even would be fully covariant to general relativity and lead to the Einstein equations in the macroscopic limit.

Although largely unknown in the scientific community, there is a theory which could be considered and analysed as a candidate for such a fundamental theory.

3 Heim's field theory

It was developed over decades by the German physicist Burkhard Heim (1925–2001),[5] but unfortunately published only once in a scientific journal in form of a compact overview article in 1977 [42], giving an outline, but no details of the theory and its results. Later, in the 80s, Heim published these details in two books ([43, 44]), written in German language, in a publishing house non-standard for physics. This led to the unfortunate situation that his theory in the elaborated form never was peer-reviewed and not noticed by the scientific community. During the last two decades the theory was taken up by W. Dröscher[6] and J. Häuser, but redrafted to a so-called 'Extended Heim Theory' (EHT, [45–49]), which only has the poly-metric approach (see Subsection 3.5) in common with the original theory. They focus on the research for novel gravity-like interactions and exotic particles, which could be the basis for revolutionary new propulsion concepts, respectively, explain the puzzles of dark matter and dark energy. In our understanding, the EHT tries to define a mapping of its poly-metric structure to the known interactions and to find the form of two further gravity-like interactions. As far as we can see, it does not provide a new approach to calculate a mass spectrum of elementary particles, and its authors (meanwhile) evaluate Heim's mass formula [44] as incorrect [49].

Nonetheless, the author thinks that Heim's original theory is still worthwhile to be considered, as it can meet some first principle requirements for a suitable fundamental theory mentioned above (see also Appendix J) and since it provides a mathematical model with a non-linear field equation which attempts to describe the physics of spacetime and matter in a unified geometric approach ("Einstein's dream"). With its explicit non-linearity it fulfils the pre-requisite for a "true" theory, as argued in the previous section.

However, a huge amount of work remains to be done to analyse whether and how the achievements of quantum field theory and the SM could be derived from the theory, i.e. how a linkage between the theories could look like. This paper, beside reporting the achieved concrete results (in Section 4), also pursues the objective to draw attention to Heim's work and to encourage a scientific analysis and debate on his theory, by giving a hint with the subsequent results that relevant insight could be gained out of it.

Therefore, we give a sufficiently detailed outline of Heim's theory to our best ability in this Section 3, thereby transferring some details and longer calculations into several appendices. We explicitly mention where we add our own amendments or adaptations. We hope that the theory, i.e. its parts presented here being relevant for our

4 Note that also Heisenberg's famous (although not successful) spinor-"world"-equation was non-linear [39].

5 Heim first studied chemistry and then physics in Göttingen and got his diploma in physics from C. F. v. Weizsäcker in 1954. For details of his life see [40, 41].

6 A collaborator of Heim since the 80s. More precisely, an enhanced version of the theory with 8 dimensions, put forward by Dröscher, was taken up, see Appendix C.

Table 1: Calculation of rest energies (masses) in MeV for a selected subset of particles: Column 2 contains the empirical masses, E_1 and E_2 are calculated according to Eq. (70), E_3 according to Eq. (71), all three with parameters $f_{ext} = -2.1573, s = 1.089$.

						Calculated Mass Spectrum:							
Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
Leptons:													
e	0.511	1	0	1	1	0.511	0.00	0.511	0.00	0.511	0.00	0.511	0.00
μ	105.66	1	2	3	3	106.26	0.57	106.43	0.73	104.51	1.09	105.04	0.59
τ	1776.86	1	2	51	51	1779.65	0.16	1778.60	0.10	1776.70	0.01	1785.65	0.49
Quarks:													
u, d	315.00**	1	2	9	9	316.16	0.37	315.44	0.14	313.54	0.46	315.11	0.04
s	525.00**	1	2	15	15	525.34	0.07	524.46	0.10	522.56	0.46	525.19	0.04
c	1270.00	2	2	18	36	1254.14	1.25	1254.14	1.25	1254.14	1.25	1260.46	0.75
b	4180.00	2	2	60	120	4180.47	0.01	4180.47	0.01	4180.47	0.01	4201.52	0.51
Mesons:													
$\pi^{\pm}$	139.57	2	2	2	4	139.35	0.16	139.35	0.16	139.35	0.16	140.05	0.34
π^0	134.98	2	2	2	4	139.35	3.24	139.35	3.24	139.35	3.24	140.05	3.76
$K^{\pm}$	493.68	2	2	7	14	493.84	0.03	491.53	0.43	487.72	1.21	490.18	0.71
K^0	497.61	2	2	7	14	493.84	0.76	491.53	1.22	487.72	1.99	490.18	1.49
η	547.86	2	2	8	16	557.40	1.74	557.40	1.74	557.40	1.74	560.20	2.25
$\rho(770)$	775.26	2	2	11	22	772.51	0.36	770.22	0.65	766.42	1.14	770.28	0.64
$\omega(782)$	782.65	2	2	11	22	772.51	1.30	770.22	1.59	766.42	2.07	770.28	1.58
$K^*(892)$	894.24	2	2	13	26	911.85	1.97	909.57	1.71	905.77	1.29	910.33	1.80
$\eta'(958)$	957.78	1	2	27	27	943.50	1.49	942.51	1.59	940.61	1.79	945.34	1.30
$f_0(980)$	990.00	2	2	14	28	975.44	1.47	975.44	1.47	975.44	1.47	980.36	0.97
$a_0(980)$	980.00	2	2	14	28	975.44	0.46	975.44	0.46	975.44	0.46	980.36	0.04
$\phi(1020)$	1019.46	1	2	29	29	1013.18	0.62	1012.18	0.71	1010.28	0.90	1015.37	0.40
Baryons:													
p	938.27	1	2	27	27	943.50	0.56	942.51	0.45	940.61	0.25	945.34	0.75
n	939.57	1	2	27	27	943.50	0.42	942.51	0.31	940.61	0.11	945.34	0.61
$N(1440)$	1440.00	1	2	41	41	1431.26	0.61	1430.23	0.68	1428.33	0.81	1435.52	0.31
$N(1520)$	1515.00	1	2	43	43	1500.94	0.93	1499.90	1.00	1498.00	1.12	1505.55	0.62
$N(1535)$	1530.00	2	2	22	44	1532.84	0.19	1532.84	0.19	1532.84	0.19	1540.56	0.69
Δ	1232.00	1	2	35	35	1222.22	0.79	1221.20	0.88	1219.30	1.03	1225.44	0.53
$\Delta(1600)$	1570.00	1	2	45	45	1570.62	0.04	1569.58	0.03	1567.68	0.15	1575.57	0.35
$\Delta(1620)$	1610.00	2	2	23	46	1608.57	0.09	1606.31	0.23	1602.51	0.46	1610.58	0.04
Λ	1115.68	2	2	16	32	1114.79	0.08	1114.79	0.08	1114.79	0.08	1120.41	0.42
$\Lambda(1405)$	1405.10	2	2	20	40	1393.49	0.83	1393.49	0.83	1393.49	0.83	1400.51	0.33
$\Lambda(1520)$	1519.00	2	2	22	44	1532.84	0.91	1532.84	0.91	1532.84	0.91	1540.56	1.42

Table 1: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
						Calculated Mass Spectrum:							
$\Lambda(1600)$	1600.00	2	2	23	46	1608.57	0.54	1606.31	0.39	1602.51	0.16	1610.58	0.66
Σ^+	1189.38	2	2	17	34	1190.54	0.10	1188.27	0.09	1184.47	0.41	1190.43	0.09
Σ^0	1192.64	2	2	17	34	1190.54	0.18	1188.27	0.37	1184.47	0.69	1190.43	0.19
Σ^-	1197.45	2	2	17	34	1190.54	0.58	1188.27	0.77	1184.47	1.08	1190.43	0.59
$\Sigma(1385)^+$	1382.80	2	2	20	40	1393.49	0.77	1393.49	0.77	1393.49	0.77	1400.51	1.28
$\Sigma(1385)^0$	1383.70	2	2	20	40	1393.49	0.71	1393.49	0.71	1393.49	0.71	1400.51	1.21
$\Sigma(1385)^-$	1387.20	2	2	20	40	1393.49	0.45	1393.49	0.45	1393.49	0.45	1400.51	0.96
Ξ^0	1314.86	2	2	19	38	1329.88	1.14	1327.62	0.97	1323.82	0.68	1330.48	1.19
Ξ^-	1321.72	2	2	19	38	1329.88	0.62	1327.62	0.45	1323.82	0.16	1330.48	0.66
$\Xi(1530)^0$	1531.80	2	2	22	44	1532.84	0.07	1532.84	0.07	1532.84	0.07	1540.56	0.57
$\Xi(1530)^-$	1535.00	2	2	22	44	1532.84	0.14	1532.84	0.14	1532.84	0.14	1540.56	0.36
$\Omega(1672)^-$	1672.45	2	2	24	48	1672.19	0.02	1672.19	0.02	1672.19	0.02	1680.61	0.49
Gauge Bosons:													
m_{gb}	42891.7#	2	4	9	18	43,591.2	1.63	43,270.9	0.88	42,750.4	0.33	43,182.0	0.68
W	80,379.0	2	4	17	34	81,582.8	1.50	81,269.8	1.11	80,750.7	0.46	81,566.0	1.48
Z	91,187.6	2	4	19	38	91,081.9	0.12	90,769.7	0.46	90,250.8	1.03	91,162.0	0.03
Average error Δ:							0.67		0.68		0.75		0.76
Ratio with 'statistical' error Δ/Δ_s:							0.52		0.53		0.58		0.59

E_4 according to Eq. (71) et sqq. with $\exp(\pi a) := 1/2\alpha = 68.518$, i.e. an approximation which corresponds to the phenomenological approach of Section 2. The error columns display the percentaged discrepancies between calculation and empirical mass. *: For even terms ($N = 2n$) always $C = 0$ holds (no contribution of $F_{(3)}$). **: Masses in the constituent quark model. #: 0.5 $(m_W + m_Z)/2 \approx$ Mac Gregor's gauge boson mass unit $m_{gb} = m_{u,d}/\alpha$.

content, the mass spectrum of elementary particles, so becomes understandable and reviewable. Other parts of Heim's theory which lie more in the area of gravitation and cosmology and result, amongst others, in an adjusted Newtonian law (compare the MOND theory [50]), in a derived global length scale and a minimum particle mass, can only be sketched in a nutshell in Appendix M, in order to specify the minimum mass. An English written summary of Heim's overall theory, nearly without mathematics however, can be found in [51].

3.1 Foundations

Since Albert Einstein, with the theory of General Relativity (GR), found a theory which explains the phenomena of gravitation by a geometrical solution, there have been attempts to explain all physics, i.e. the elementary foundations of physics, by a generalised geometrical approach. Einstein himself tried to find a theory which not only describes gravitation, but also electromagnetism in a unified geometrical way, but did not succeed. String theory is another geometric approach, up to now without a breakthrough to concrete results comparable to experimental data. As we shall see in the following, Heim's theory is based on a geometrised field in a 6-dimensional spacetime, which is quantised on a microscopic scale and converges to GR and the Einstein field equation on the macroscopic large scale.

B. Heim was led by the following thoughts as foundations of his theory [43]. There are empirically well-based principles for a mathematical description of the physical world:

a. Validity of general conservation laws as for energy, momentum or electric charge.
b. There are extremum principles (as e.g. for entropy with the second law of thermodynamics) which can be formulated by variation theorems.
c. Physical action is always experienced as whole-number multiples of h, Planck's constant, the minimal action. The atomic structure of matter and non-existence of an energetic continuum are consequences of the quantum principle.
d. We experience the interaction fields of electromagnetism (EM), of gravitation and of the strong and weak interactions of short range, where the (classical) electromagnetic field is described by Maxwell's equations in a gauge invariant way (d1), gravitation by GR, which in normal cosmic distances leads to Newton's law, not gauge-invariant (d2). The short-range interactions act in nuclear scale. Our laws for these interactions describe nature within the range of accuracy of measurement, but do not have to be finally exact.

The experiments of high-energy physics have brought a broad spectrum of particles which can be classified by a set of quantities (quantum numbers) like charge, baryon number, spin, isospin, strangeness etc. Our theories, first of all the SM, make determinations about the question which particles are elementary, i.e. not further dividable, but at least in the case of the strong interacting hadrons it is still difficult to judge whether mesons and baryons are elementary (not dividable) or the quarks (and gluons).

But all particles have the property of mass as a measure for inertia and, with the principle of equivalence between inert and heavy mass, of gravity, and with Einstein's equivalence of mass and energy $E = mc^2$, of energy as well. These principles are the logical starting point for Heim for a unified description of matter. Heim distinguishes between ponderable particles which have rest mass and imponderable particles without rest mass like the photon. Heim's goal was to describe all particles in a unified manner, i.e. in a unified theory and to calculate the masses of the particles.

Due to the equivalence principles, each particle with or without rest mass, through its energy, must be considered as a source of gravitation. Therefore, gravitation is the general background phenomenon which belongs to all particles as basis of matter, and Heim starts by analysing the role of gravity in a purely phenomenological way. He recognises that each piece of matter creates, beside its rest mass as source of a gravitational field, an energy-mass of this field $\left(\mu = E_{\text{field}}/c^2\right)$ which is a source of gravitation, too (although very much smaller than the rest mass). Heim found equations for the relation between rest mass density σ, field mass μ, the gravitational field ($\vec{G} = \text{grad}\ \phi$) and an introduced "meso field" $\vec{\mu} = a\dot{\vec{G}} + \sigma\vec{v}$, with $\vec{v}$ being the velocity of the moving mass, which have similarities with Maxwell's equations (see [43]). The source of the gravitational field and the generated field build a unity.

3.2 Non-Hermitian spacetime structure

Putting together these phenomenological fields of gravity and the Maxwell theory, the resulting unified field tensor and, consequently, the respective energy momentum tensor turn out to be non-hermitian $T_{ik} \neq T^*_{ki}$. In a geometric description, considering the 4 interactions, to be separated into gauge invariant and non-gauge invariant interactions with at least the gravitation being of the second type, one can derive that a general (geo)metric tensor g_{ik} (as coefficients of a homogeneous quadratic differential form) in the R_4 is asymmetrical as well, see Appendix A for details. The symmetric part of g_{ik}, $g_{ik}^{(1)}$, resulting from the non-gauge invariant interaction (→Gravity), alone makes up the resulting Riemannian geometry $g_{ik}^{+} = \left(g_{ki}^{+}\right)^{*}$ as $g_{ik}^{-} = -\left(g_{ki}^{-}\right)^{*}$, so $g_{ik}^{-}\mathrm{d}x_i\mathrm{d}x_k = 0$.

However, there can be parallel shifts irrespective of the metric, and the corresponding Christoffel symbols remain non-hermitian and split into a hermitian and non-hermitian part $\Gamma_{km}^{i} \neq \Gamma_{mk}^{i*}$.

So, a covariant differentiation becomes possible and a curvature tensor R_{kmp}^{i} can be defined, being non-hermitian as well. This is valid for its Ricci Tensor $R_{kmi}^{i} = R_{km} \neq R_{mk}^{*}$, too.[7] Further theorems and identities can be derived for $g_{ik} \neq g_{ki}^{*}$ which are not analogous to Riemannian geometry, while the geodesic equation

$$\ddot{x}^i + \Gamma_{km}^{i}\dot{x}^k\dot{x}^m = 0 \qquad (12)$$

is formally the same as in the hermitian case. With $x_4 \sim t$ in the R_4 the $\dot{x}$ are accelerations which are always caused by interactions, so that the $g_{ik} \neq g_{ki}^{*}$ should be identified as non-hermitian interaction potentials in tensor form. If we now approximately set the other contributions of g_{ik}, $g_{ik}^{(2)} = g_{ik}^{(3)} = 0$ (see Appendix A), so $g_{ik} = g_{ik}^{(1)} = g_{ki}^{(1)*}$, we get the Riemannian geometry. If we further reduce the phenomenological energy momentum tensor T_{ik} to the one of an electromagnetic field and classical point particles, i.e. neglect the gravity field part so that $T_{ik} = V_{ik} = V_{ki}^{*}$ becomes hermitian, then the divergence of V_{ik} is 0 due to the known conservation laws of EM and classical physics.

In GR the $g_{ik}^{(1)}$ of the Riemannian geometry, due to the geodesic equation, are interpreted as tensor potentials of the gravitational field. The only possible divergence-free structure tensor, depending on $g_{ik}^{(1)}$ and its first and second partial derivatives, is $R_{ik}^{(1)} - \frac{1}{2}g_{ik}^{(1)}R^{(1)}$, which is set $\sim V_{ik}$:

$$R_{ik}^{(1)} - \frac{1}{2}g_{ik}^{(1)}R^{(1)} \sim V_{ik} \qquad (13)$$

This fundamental relation, the Einstein equation, is so interpreted in GR that the phenomenological tensor V_{ik}, being proportional to the divergence-free structure tensor, as source of gravity generates this structure field of Riemannian geometry, which on its part is interpreted as the gravitational field. Heim's postulate is that, due to the validity of GR in the macroscopic realm, each further theoretical approach must converge to the Einstein equation in a respective approximation. Following the scheme of (13), Heim now relates the whole non-hermitian structure tensor to the whole phenomenological tensor T_{ik}:

$$R_{ik} - \frac{1}{2}g_{ik}R \sim T_{ik} \qquad (14)$$

In absence of the non-symmetric parts in g_{ik} and T_{ik} this equation turns into the GR equation (13). But in general, both sides of (14) are not necessarily divergence-free, so that the conservation laws might be violated, which shall be tolerated as a start.[8] Therefore, according to Heim, relation (14) should be interpreted this way:

It expresses the validity of equation (13) and the overall *equivalence* between two sides which both incorporate the whole physics, the non-hermitian structure part on the left side and the also non-hermitian phenomenological part described by T_{ik} on the right side, which already contains the phenomenological gravity with its field and source (see above). The non-hermitian tensor g_{ik} is to be considered as the tensorial potential of interactions in the R_4, which can be seen from the geodesic equation.

As pointed out by Heim in his first book [43], Einstein himself undertook an attempt to bring the hermitian metric tensor of his theory of GR into a more general non-hermitian form by adding a speculative anti-hermitian term with the purpose to eliminate the phenomenological tensor from GR, see Einstein's book "The meaning of Relativity" from 1954 (fifth edition with further appendix, [53]). But the so created non-hermitian structure relations of a Weyl-like spacetime geometry seem to have a too limited manifold of solutions as it could be a universal law of nature [43]. Furthermore, this approach, as GR, excludes the whole empirical realm of quantum physics (see c.).

3.3 Quantised nature and world dimensions

Also the theoretical ansatz (14) does not yet account for the quantum nature. It can be introduced in a phenomenological manner by the following thoughts:

Equation (14) can be brought into the form $R_{ik} \sim T_{ik} - \frac{1}{2}g_{ik}T = W_{ik}$ with T being the trace of T_{ik} and W_{ik} the components of a different form of energy momentum tensor. These energies relate to the R_3 and are the time derivatives of actions ω_{ik}. For the W_{ik} then $W_{ik} \sim c\frac{\mathrm{d}\omega_{ik}}{\mathrm{d}\Omega}w$ holds with $\mathrm{d}\Omega = w\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$, $x_4 = ct$ (or ict in Minkowskian metric) and $w = \sqrt{-\det\ g_{ik}}$, the functional

7 Using the notation of Heim for the R_{kmp}^{i} which differs from the notation, e.g. in [52] by an exchange of the last two indices.

8 It will be seen later that violating parts cancel out, see Sections 3.4 and 3.5.

determinant. According to nature (principle c.) generally $\omega_{ik} = hN_{ik}$ holds, where the N_{ik} in general are complex numbers with integer real and imaginary part. Due to this integrity, the typical discontinuity appears in microscopic quantum levels. This means that the differentials in W_{ik} have to be exchanged by differences. With the density η_{ik} defined by $\eta_{ik}\Delta\Omega = \Delta N_{ik}$, the W_{ik} read $W_{ik} \sim chw\eta_{ik}$, or

$$R_{ik} \sim w\eta_{ik}. \tag{15}$$

η_{ik} expresses the number of quanta of action per spacetime volume. Relation (15) shows that due to principle c. the structure R_{ik} with Γ^i_{km} and g_{ik} does no longer appear as metric continuum in the microscopic realm, but in discrete quantised steps. On the other hand, the difference $\Delta\Omega$ in the discontinuous density η_{ik} indicates that spacetime itself may be discontinuous, probably by the existence of a smallest geometric unit.

From the discretisation of the structure field (this field being a radical geometrisation of the phenomenology) follows that the spacetime R_4 must be considered as a medium with a Hilbert functional space, i.e. a convergent state function (field) ϕ^i_{km} of the metric state of spacetime must exist and a hermitian state operator C_p, acting on ϕ^i_{km} in such a manner that an equivalent to the metric structure term arises:

$$C_p\phi^i_{km} \to C_p\Gamma^i_{km} = R^i_{kmp} \tag{16}$$

with Γ^i_{km} being the Christoffel symbol and R^i_{kmp} the curvature tensor in the macroscopic realm.

On the other hand, because of the convergence of the state function and its hermiticity, this operator must define a spectrum of eigenvalues λ_p

$$C_{(p)}\phi^{(p)}_{km} = \lambda_{(p)}(k,m)\phi^{(p)}_{km} \tag{17}$$

which[9] are proportional to energy densities, as the contracted macroscopic curvature tensor ($i = p$) is proportional to energy densities as well. These eigenvalues form a discrete point spectrum and give possible states of a microscopic field source (a more detailed step-by-step derivation of equation (17), according to [43], is given in Appendix B). In this system of tensorial operator equations the 3 indexes run independently over the 4 spacetime dimensions, i.e. there are 64 discrete eigenvalue spectra of metric structure. As this system of eigenvalue equations is non-linear (due to C_p), the $\left|\phi^{(p)}_{km}\right|^2$ cannot be interpreted as probabilities, since the solutions of the state functions of the metric structure do not additively superpose. So, the C_p describe non-linear metric states of a non-hermitian spacetime which occur in metric levels $\lambda_p(k,m)$. On the other hand, these $\lambda_p(k,m)$ are equivalent to the energy masses of those physical elementary structures which, according to Heim, appear as spacetime deformations in the sense of empirical elementary particles (see Subsection 3.4.5 and Section 4). Due to the non-linearity of the spacetime states the theoretical masses will not have quantum theory-like uncertainties, but will have discrete values in accordance with the results of empirical high-energy physics.[10]

In the macroscopic realm the discrete density η_{ik} of (15) becomes a steady function $w\eta_{ik} \to T_{ik} - \frac{1}{2}g_{ik}T$ and, with contraction of p and λ_p depending on k and m, $\lambda_p = \lambda_p(k,m)$, (17) turns into

$$C_p\phi^p_{km} \to R_{km} \quad \text{and} \quad \lambda_p\phi^p_{km} \to \kappa\left(T_{km} - \frac{1}{2}g_{km}T\right), \tag{18}$$

i.e. the Einstein equation is obtained again. Because of the correspondence (16) the algebraic properties of the curvature tensor are transferred to $C_p\phi^i_{km}$, so from

$$\begin{aligned} R^i_{kmp} &= \Gamma^i_{kp,m} - \Gamma^i_{km,p} + \Gamma^i_{ms}\Gamma^s_{kp} - \Gamma^i_{ps}\Gamma^s_{km} \\ \Rightarrow\quad C_p\phi^i_{km} &= \phi^i_{kp,m} - \phi^i_{km,p} + \phi^i_{ms}\phi^s_{kp} - \phi^i_{ps}\phi^s_{km} \end{aligned} \tag{19}$$

follows, with $(), m := \partial_m = \partial/\partial x_m$. For $m = p$, $C_{(m)}\phi^i_{k(m)} = 0$ holds and therefore $\lambda_{(m)}\phi^i_{k(m)} = 0$ and thus[11,12]

$$\lambda_m(k,m) = \lambda_m(m,k) = 0. \tag{20}$$

So, beside the 16 spectra, given by $\lambda_m(k,m)$, further 16 spectra $\lambda_m(m,k)$ are generally empty, but the four spectra $m = k$, $\lambda_m(m,m) = 0$ appear twice. This reduces the number of empty spectra from 32 to 28. These 28 in principle empty spectra must trace back to the nature of

9 The brakets mean no summation over this index.

10 The author considers it as an interesting open question whether or how an analogy could be drawn between the appearance of non-linearity given here for quantities which describe spacetime and matter in a geometric approach, and the non-linearity in Quantum Field Theory (QFT), given already in the QED (see e.g. [2]) or in the Dyson Schwinger equations. Both produce difficulties in applying the standard rules of quantum mechanics. A linearisation always seems to be necessary to achieve probability statements, but produces infinities in the case of QFT (when the perturbation series are considered as linearisation).

11 The ϕ^i_{km} vanish only in the Euclidean case or for geodesic coordinates, but are in general $\neq 0$, describing the deviation from the Euclidean metric. The 64 possible eigenvalue spectra $\lambda_p(k,m)$ are discrete spectra of structure levels (equivalent to energetic states) and so still unknown functions of integer quantum numbers. It is possible that despite $\phi^i_{km} \neq 0$ some of these spectra in principle remain empty, so that $\lambda_{(m)}(k,m)\phi^i_{k(m)} = 0$ in general is fulfilled through $\lambda_m(k,m) = 0$.

12 The $\lambda_p(k,m)$ are symmetric and real due to the hermiticity in (17).

the non-hermitian spacetime structure itself. As the λ_p determine the discontinuous structure field of spacetime with the property (20), the 64 relations of equation (17) must be substituted with the 28 of (20), so $64 - 28 = 36$ discontinuous energy momentum densities $\lambda_{(p)}\phi^{(p)}_{km} \sim \epsilon^{(p)}_{km} \neq 0$ remain. These densities must be invariant against the allowed coordinate tranformations, i.e. they must be components of a canonical energy momentum density tensor, i.e. a second degree tensor whose zero divergence expresses principle a. The 36 components can be placed in a 6-dimensional quadratic matrix scheme. Such a 6-dimensional density tensor can only be constituted in a space whose dimension is identical with the row number of the tensor. Because rows and columns of a tensor are always vectors.

With this line of thought Heim (in [43]) deduces that the R_4 should be extended by two hidden dimensions x_5 and x_6 to a R_6 in such a way, that the R_4 spacetime becomes a subspace of this R_6. Heim analyses the general condition for describing components of quadratic tensors of a lower dimensional space in a higher dimensional one, each with whole-number dimensions, and derives a respective formula, see Appendix C.

Now, the $\epsilon^{(p)}_{\alpha\beta}$ with indices α, β of the R_4 have to be related to the tensor T_{ik} in the R_6, $i, k = 1$ to 6, $\epsilon^{(p)}_{\alpha\beta} \hat{=} T_{ik}$. This assignment cannot be arbitrary, because the spacetime sector $T_{\alpha\beta}$ must match the phenomenological term (right side) of (14), while T_{ik} arises from this spacetime section by a double bordering with T_{i5} and T_{i6}, respectively, T_{5i} and T_{6i}. If j denotes the indices 1 to 3 of the R_3 of physical space, then the space-like components of this bordering T_{j5} and T_{j6} with their transpositions would mean that x_5 and x_6 structures would directly influence those of the R_3 if $T_{j5} \neq 0$ and (or) $T_{j6} \neq 0$. In this case typical physical phenomena would have to be empirically observable which, however, never have been detected by experimental physics. For this empirical reason the 12 space-like components are to be set $T_{j5} = T_{j6} = 0$ and $T_{5j} = T_{6j} = 0$.[13]

Thus, the tensor scheme of the R_6 can be determined as follows:

$$T_{ik} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} & 0 & 0 \\ T_{21} & T_{22} & T_{23} & T_{24} & 0 & 0 \\ T_{31} & T_{32} & T_{33} & T_{34} & 0 & 0 \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ 0 & 0 & 0 & T_{54} & T_{55} & T_{56} \\ 0 & 0 & 0 & T_{64} & T_{65} & T_{66} \end{pmatrix} \tag{21}$$

Heim constructs the R_6 (as medium space for the Hilbert space) with a signature $I(+++---)$.[14] Different than in Kaluza Klein theories and in String theory the additional coordinates are not compactified, but get physical meaning. They cannot be two additional times, neither can they act on the movement of point particles in three-dimensional space, because in spaces with more than three dimensions, e.g. calculations of planet orbits lead to spiral orbits which are not observed. (The third possibility of mixed signs in the signature of the additional coordinates is excluded as this would lead to a different sign in the functional determinant $w = \sqrt{-g}$ which should remain unchanged compared to the R_4 as the space is only extended.)

Therefore, Heim identifies these additional coordinates as parameters with organisational and informational effects which only act on structures, but not on points and their paths. They are considered as true world dimensions.[15]

It becomes apparent that the tensor components of the 'trans'-sector, T_{55}, T_{56}, T_{65} and T_{66}, can influence the spacetime sector only via the time sequences T_{i4} and T_{4i} with $1 \leq i \leq 6$ and so only indirectly the R_3. As the coordinates x_5 and x_6 cannot (directly) be measured

13 In [43], Heim gives a derivation for this determination which had been found by his collaborator W. Dröscher: Due to the transition $C_p\phi^i_{km} \to R^i_{kmp}$ into the macroscopic realm, independently from the symmetry properties of the g_{ik}, but only from the algebraic structure of (17) the symmetry $C_p\phi^i_{mp} = -C_m\phi^i_{pp}$ or $\lambda_p(m,p)\phi^i_{mp} + \lambda_m(p,p)\phi^i_{pp} = 0$ follows. Herein the symmetry of the $C_p\phi^i_{mp}$ directly results from that of the R^i_{kmp}, for which $R^i_{ppm} + R^i_{pmp} = 0$ holds (using the notation of Heim for the R^i_{kmp}, which differs from the notation, e.g. in [52] by an exchange of the last two indices), and from the symmetry of the ϕ^i_{mp} in the lower indices. As $\phi^i_{km} \neq 0$ always holds if a metric structure is presumed and also $\left|\phi^i_{km}\right| < \infty$, but always $\lambda_m(m,p) = \lambda_m(p,m) = 0$ applies (20), the identities $\lambda_m(p,p) = 0$, respectively, $\lambda_p(m,m) = 0$ follow from the symmetry relation above. 4 of these 16 relations were already taken into account in (20) with $p = m$. So, 12 further vanishing eigenvalue spectra remain, $\lambda_p(m,m) = 0, p \neq m$, which means (because of $\epsilon^{(p)}_{\alpha\beta} \hat{=} T_{ik}$) that the tensor T_{ik} of the R_6 indeed contains 12 components which vanish. These zeroes can neither lie in the spacetime sector nor in the temporal bordering so that the empirical requirement $T_{j5} = T_{j6} = 0$ and $T_{5j} = T_{6j} = 0$ is fulfilled and explained.

14 The analogue in the R_4, $I(+++-)$, is defined by a metric tensor g_{ik} which, in an Euclidean or local geodesic space, becomes the diagonal η_{ik} with $ds^2 = \eta_{ik}dx_i dx_k = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = dr^2 - c^2dt^2$. The $I(+++---)$ means in an analogous way $ds^2 = dr^2 - c^2dt^2 - dx_5^2 - dx_6^2$ in a geodesic R_6 space.

15 With regard to the $R_4 x_5$ and x_6 have the properties of hidden world coordinates which should not exist due to the Kopenhagen interpretation of quantum mechanics. But possibly, the restriction to the four spacetime dimensions and exclusion of a hyperspace was a too narrow concept.

and the T_{ik} are microscopic quantities, this means that the future is always open, i.e. predications about the future can only be statements of probability.[16] Only if many micro states superpose and build a macroscopic collective system, a uniqueness of factual causality is "pretended".

The extended Cartesian coordinate system then reads in Minkowskian notation (the − signatures in η_{ik} lead to $\sqrt{-1} = i$ factors):

$$(x_1, x_2, x_3) \quad \hat{=} R_3, \quad x_4 = ict, \quad x_5 = i\epsilon, \quad x_6 = i\eta. \quad (22)$$

These 6 coordinates are independent of each other, so a normalised orthogonal system exists with unit vectors:

$$1 \leq k \leq 6, \quad \vec{x}_k = \vec{e}_k x_k, \quad \vec{e}_i \cdot \vec{e}_k = \delta_{ik} \quad (23)$$

The structure of (21) suggests that the set of the R_6 coordinates is structured in the form $\{x_1, x_2, x_3\}$, $\{x_4\}$, $\{x_5, x_6\}$. The fundamental eigenvalue Eq. (17), according to Heim, now becomes valid for the R_6, i.e. with indices $p, k, m = 1, \ldots, 6$ and the discrete eigenvalues $\lambda_p(k, m)$ being proportional to energies. It describes all elementary structures of matter.

At the end of this paragraph we shortly want to discuss[17] Heim's finding of a smallest element of area, which he was able to derive from his enhanced phenomenological theory of gravity which takes into account the field mass (compare above) and leads to a deviation from Newton's law for very large distances (see Appendix M, subsection M.1, and [42, 43])

$$\tau = \frac{3}{8}\frac{\gamma h}{c^3} \approx 6,15 \cdot 10^{-70}\mathrm{m}^2 \quad (24)$$

with γ being the gravitational constant, h Planck's constant and c the velocity of light. This smallest area, which was also identified by Treder [56], is proportional to the square of the Planck length and was called *metron* by Heim. Heim was the first who concluded from this detection that the smallest area makes a new calculus with differences of these small but finite areas necessary. Contrary to GR and Quantum Field Theory, infinities and singularities cannot occur in the metron calculus. Heim developed this calculus completely on his own (see [43]), although there were already studies on calculus with differences ([57–59]).

16 Heim reminds at this point that C. F. v. Weizsäcker ingeniously deduced the whole abstract quantum mechanics as framework from the existence of separable alternatives and from the existence of an open future [54]. The second premise obviously is fulfilled in Heim's 6-dimensional theory from the start.

17 We follow the summary in [55].

The infinitesimal tensorial eigenvalue relationship (17), which described metric structural levels in the continuous R_6, is translated by Heim into a discrete R_6, with tensors becoming so-called selectors and curvatures of space becoming so-called condensations (of metrons). Under spacetime curvatures one has to imagine compressions or condensations of the smallest areas of the discontinuous world continuum, when projected onto Cartesian reference areas. The correspondences to the Christoffel symbols act as condensers. Therefore, instead of curvatures, in the following the notion condensation is used. To keep a comparison with Heim's original texts ([43, 44]) easy, we also use the terms selector and condenser, but occasionally put the respective term of Riemannian geometry and GR in brackets behind (tensor etc.).

In Heim's findings on cosmology [44] the metron appears cosmologically as a very slowly decreasing scalar function of the world age. As the cosmos expanded, the metrons kept dividing. A point in time can be determined when in the past a metron was so big that its surface encompassed the entire cosmos. Instead of a big bang in Heim's cosmology the first division of the metron occurred, which then continued during the expansion of space. For a very long time, spacetime was embossed only due to this structural dynamic. Only as the fluctuations in densities and metrons and their exchange in the sub-rooms of the R_6 became more numerous, what we call energy and matter was formed.

As Heim's detailed theory of gravity and cosmology shall not be subject-matter of this article, the metron theory need not be introduced, because in relation to three-dimensional space, there are four areas of validity of the components of the fundamental condenser:

1. a "metronic area" in which the number of metrons is relatively small,
2. an area with a high number of metrons,
3. an "infinitesimal area" in which the number of metrons is so large that the structural (spacetime) quantisation can be neglected, but quantum levels exist, and
4. a "macroscopic area" in which the fundamental condenser exponentially strives for a constant fixed value.

The microphysical processes which we consider in this paper are empirically given only in the third area of validity (3.). Therefore, by default the metron calculation can be dispensed with and the normal tensor algebra and calculus can be applied. However, we shall see that when it comes to the calculation of energy by means of a spatial volume element in our formalism, the quantisation of space (via the metrons) is to be applied – to account for quantisation of energy (see Section 4 and Appendix K). In treating the evolution of quantum cosmology in Heim's theory the use of the metron picture is essential.

3.4 Solutions of Heim's 6-dimensional field equations in the microscopic realm

3.4.1 The Hermitian R_6 and fundamental equations

As a first step towards solving the 6-dimensional field equations, we have to make clear their algebraic character. Instead of the situation in the R_4, the metric tensor and the energy density tensor in the R_6 are Hermitian quantities. Heim deduces this fact from the requirement that transformations of the 6-dimensional word structures should be unique, have a unique inverse, that no singularities should appear and the coordinate transformations must be steady, meaning that the tranformations must be those of the global Poincare group in the R_6. As the world structures to be described are non-Euclidean as metric deformations of 6-dimensional manifolds, we must furthermore distinguish between co- and contravariant quantities. Due to the Poincare invariance, a transformation to geodesic coordinates η_k must exist with unit vectors $\vec{\epsilon}_k$ and $\vec{\eta}_k = \vec{\epsilon}_k\eta_k$, being not necessarily orthogonal (because of the non-Euclidean structure), so that $(\vec{\epsilon}_i\vec{\epsilon}_k)_6 = \hat{A}(x_1 \dots x_6) \neq \hat{E}$. With $\hat{A} = \hat{A}_T$ and the geodesic η_k and their total differential relative to arbitrary coordinates x_i of the R_6 it follows

$$\mathrm{d}s^2 = \sum_{p,q=1}^{6} \mathrm{d}\vec{\eta}_p \mathrm{d}\vec{\eta}_q^* = \sum_{p,q=1}^{6} \frac{\partial\vec{\eta}_p}{\partial x^i}\frac{\partial\vec{\eta}_q^*}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k = g_{ik}\mathrm{d}x^i\mathrm{d}x^k = g^{ik}\mathrm{d}x_i\mathrm{d}x_k. \tag{25}$$

Here the coefficients $g_{ik} = \vec{\eta}_{,i}\vec{\eta}_{,k}^* = g_{ki}^*$ are hermitian. So, the world structures of the R_6 with the metric tensor g_{ik} are hermitian, although those of the subspace R_4 were non-hermitian. This might be interpreted by the fact that the R_4 structures are only spacetime 'slices' of the general R_6 world structures. The relation (25), finally being a postulate for the R_6, shows a higher symmetry than the respective relation for the R_4 (see Appendix A).

Because of the hermiticity of g_{ik} all theorems and identities of Riemannian geometry hold in the so defined R_6, where the parallel shifts are described by Christoffel symbols in the well-known way $\Gamma^i_{km} = \Gamma^{i*}_{mk}$, the metric determinant $g = |g_{ik}|_6$ (and so the functional determinant $w = \sqrt{-g}$) keeps the same sign as in R_4, as already stated, and the known theorem of Riemannian geometry $\frac{\partial g}{\partial x^k} = 2g\Gamma^m_{km}$ is valid in R_6 as well. Independent from the number of dimensions the curvature tensor can be expressed by the Γ^i_{km} as in (19). As also the empirical principle b. must be met in the R_6, the valid geodesic equation (12) tells that all accelerations in R_6 are determined by the Γ^i_{km} and so by the g_{ik}, meaning that the components of g_{ik} can be considered as universal tensorial interaction potentials and all phenomenological interactions should be describable through them, which would correspond to a radical geometrisation of physics.

The 36 discontinuous energy momentum densities $\epsilon^{(\mu)}_{\alpha\beta} \neq 0$ of the R_4 form the components of an energy momentum density tensor ϵ_{ik} in the R_6 which shall be divergence-free (as a postulate) and become $\epsilon_{ik} \to T_{ik}$ in the macroscopic realm. Analogue to GR with a hermitian metric tensor, there is only one structure tensor built from g_{ik} and its first and second partial derivatives which is $R_{ik} - \frac{1}{2}g_{ik}R$, so that again the relation $R_{ik} - \frac{1}{2}g_{ik}R = \alpha T_{ik}$ (with α as proportionality factor) is obtained.

Due to consistency at the transition from microscopic to macroscopic states there must be a convergent function $\left\{ {i \atop km} \right\}$[18] in the R_6 which converges to the macroscopic Γ^i_{km}. With the same arguments as for Eqs. (16)–(19) respective equations and relations can be derived for the microscopic realm in the R_6, leading, amongst others, to the eigenvalue equation with the hermitian operator C_p

$$C_p\left\{ {i \atop km} \right\} - \lambda_p(k,m)\left\{ {i \atop km} \right\} = Q^i_{kmp} = 0, \tag{26}$$

which is stipulated for the same reason as in the R_4, compare Appendix B. The contracted form $Q^p_{kmp} = 0$ with $C_p\left\{ {p \atop km} \right\} \to R_{km}$ directly leads again to the Einstein-equation-like relation. Both equations, the non-contracted and the contracted, with the abbreviations

18 In the following we denote the state functions of the microscopic realm in the R_6, which converge to the macroscopic Γ^i_{km}, as $\left\{ {i \atop km} \right\}$, following Heim. They must not be mixed up with the Christoffel symbols of Riemannian geometry and GR. We further denote those with Γ^i_{km}.

$$\left(\left\{ {i \atop km} \right\}\right)_6 = \widehat{\{\}}, \quad C = \sum_{p=1}^{6} C_p, \quad \vec{\lambda} = \sum_{p=1}^{6} \vec{e}_p \lambda_p, \quad \text{giving}$$
$$C_p \left\{ {i \atop km} \right\} = (C\widehat{\{\}})^i_{kmp} \quad \text{and}$$
$$\lambda_p(k,m) \left\{ {i \atop km} \right\} = (\vec{\lambda} \times \widehat{\{\}})^i_{kmp} \tag{27}$$

can be written in the more compact form

$$C\widehat{\{\}} = \vec{\lambda} \times \widehat{\{\}} \quad \text{and} \quad \mathrm{Tr}\, C\widehat{\{\}} = \vec{\lambda}\widehat{\{\}}, \tag{28}$$

Tr being the matrix trace. Beside this contraction which corresponds to R^p_{kmp}, there a two further possiblities, $R^k_{kmp} := B_{mp}$ and $R^m_{kmp} = -R_{kp}$, which are analysed in Appendix D.

Considering in how far (28) fulfils or is able to fulfil the fundamental empirical prinpicles a. to d., it can be stated that (a) is met due to the hermiticity of g_{ik} and T_{ik}, (b) as the geodesic equation is fulfilled and (c) through the existence of a Hilbert space for the hermitian state operator C_p, carried by the R_6 with real discrete eigenvalues $\vec{\lambda}$. Finally, $R_{ik} - \frac{1}{2} g_{ik} R = \alpha T_{ik}$ implies through its double singular mapping into the R_4 the statement d. with its empirical forms d1 and d2 ($R_{\pm 4}$), see Subsection 3.4.5, and also Appendix J for the linear limit and mapping to classical EM (d1).

3.4.2 Connection to the phenomenology of the R_4 and partial structures

Heim performs an analysis of the structure of the energy momentum tensor T_{ik} in the R_6 and concludes that in the *macroscopic* realm T_{ik} can be composed of the physical fields of EM and gravity, $\vec{E}, \vec{H}$ and $\vec{G}, \vec{\mu}$, and of a fifth so far unknown field $\vec{K}$, which should couple EM and gravity, see Appendix E, subsection E.1. From this three-fold structuring a metric tensor can heuristically be derived, consisting of three metric partial structures $g^{(\mu)}_{ik}$ with $1 \le \mu \le 3$ in the R_6 so that $g_{ik} = g^*_{ki}$ is considered as a composition of these partial structures μ.[19] The g_{ik} on the whole, now called composition field, is hermitian, but not necessarily the single partial $g^{(\mu)}_{ik}(x_1 \dots x_6) \neq g^{(\mu)*}_{ki}$. In a part of the R_6 which is only determined by a partial structure μ, parallel shifts can be conducted with $\Gamma^i_{km(\mu)} \neq \Gamma^{i*}_{mk(\mu)}$. Now, in the microscopic realm, there should be combinations of $\left\{ {i \atop km} \right\}_{(\mu)} \neq \left\{ {i \atop mk} \right\}^*_{(\mu)}$ which go over to the macroscopic continuum as $\left\{ {i \atop km} \right\}_{(\mu)} \to \Gamma^i_{km(\mu)}$ and superpose to $\left\{ {i \atop km} \right\} = \sum_{\mu=1}^{3} \left\{ {i \atop km} \right\}_{(\mu)}$. With the same abbreviation as in (27) $\widehat{\{\}}_{(\mu)} = \left(\left\{ {i \atop km} \right\}_{(\mu)}\right)_6$ the non-hermiticity $\widehat{\{\}}_{(\mu)} = \widehat{\{\}}^+_{(\mu)} + \widehat{\{\}}^-_{(\mu)}$ gives

$$\widehat{\{\}} = \sum_{\mu=1}^{3} \widehat{\{\}}^+_{(\mu)}, \quad \sum_{\mu=1}^{3} \widehat{\{\}}^-_{(\mu)} = 0, \tag{29}$$

because $\widehat{\{\}}$ is hermitian. Under this premise the state function in (28) becomes a true mixed variant third-degree tensor field, because geodesic coordinates can only be found in respect of one partial structure so that a geodesy for the other μ is not given and so $\widehat{\{\}}$ cannot be transformed away.

The construction of the R_6 and the existence of its hermitian structures g_{ik} as well as the microscopic law (28) apparently are a consequence of the whole-number numerator in the η_{ik} of (15), i.e. the physical quantum nature (principle c.).

3.4.3 Hermetry forms

If in Eq. (28) condensations $\vec{\lambda} \neq 0$ do exist, then these $\vec{\lambda}$ need not be defined in all coordinates of the R_6, but this spectrum can be related only to a subspace V_k with $1 \le k \le 6$ in which each fundamental selector (metric tensor) deviates from unity $g \neq E$. A physical interpretation of the k-dimensional deformable subspaces represents a *herme*neutics of the respective world geome*try*. Therefore, Heim uses the term *Hermetry* to distinguish that a deviation from Euclidean geometry is given in a V_k. If k coordinates are *hermetric*, then $6 - k$ Euclidean coordinates remain *anti-hermetric*.

According to (22) the R_6 consists of three non commutable imaginary coordinates and the real R_3 which, concerning the hermetry, appears as one semantic architectural unit $s_{(1)} = R_3$ of the world. The non commutable imaginary coordinates make up three further semantic units $s_{(2)} = x_4$, $s_{(3)} = x_5$ and $s_{(4)} = x_6$. The hermetry notion $\vec{\lambda}(V_k) \neq 0$ must apply to these 4 semantic units $s_{(j)}$. The number of basically possible hermetry forms (hermetric subspaces) can be calculated as $Z = \sum_{k=1}^{4} \binom{4}{k} = 15$. Heim now argues that the units $s_{(3)}$ and $s_{(4)}$ must be hermetric in any case:

19 With respect to the two-dimensional metron it should be mentioned that the R_6 can be considered as a fully into metrons fragmented space as 6 MOD 2 = 0. Then $M = 6/2 = 3$ is the number of possible metronic partial structures, which supports the heuristic conclusion above.

From the divergence-free hermitian energy momentum density tensor T in the R_6 and $T = \mathrm{Tr}(M \times M)$, $\mathrm{div}_6 M = 0$ follows for the field tensor. In the non-hermitian R_4 a phenomenological field tensor is defined, too, whose 4-dimensional vector divergence is proportional to a four-current describing inertia and electric charge. So, from $\mathrm{div}_6 M = 0$ follows that the non-R_4 parts of the tensor and their partial derivations with respect to x_5 and x_6 describe these fundamental quantities of the physical R_3 which reduce to field components in x_5 and x_6, thus in $s_{(3)}$ and $s_{(4)}$. Heim excludes the case that only one of the units $s_{(3)}$ or $s_{(4)}$ could be hermetric, as a world with hermetric coordinates only in the R_4 seems to be impossible due to his argument above, and the possible case of an anti-hermetric R_4 ($s_{(1)}$, $s_{(2)}$) would leave all $\lambda_p(k,m) = 0$ if only $s_{(3)}$ or $s_{(4)}$, i.e. only one coordinate were hermetric. This can be seen from the relations after Eq. (30) in the next section.

On the other hand, $s_{(1)}$ and (or) $s_{(2)}$ *can* be hermetric, but don't have to. So, for them we get the possible cases "none of them", $s_{(1)}$, $s_{(2)}$ or ($s_{(1)}, s_{(2)}$) are hermetric. Combined with the always hermetric $s_{(3)}$, $s_{(4)}$ this finally provides the following 4 remaining hermetry forms:

$$a \mathrel{\widehat{=}} H(s_{(3)}, s_{(4)}) = H(\kappa_{(1)})$$
$$b \mathrel{\widehat{=}} H(s_{(2)}, s_{(3)}, s_{(4)}) = H(\kappa_{(1)}, \kappa_{(2)})$$
$$c \mathrel{\widehat{=}} H(s_{(1)}, s_{(3)}, s_{(4)}) = H(\kappa_{(1)}, \kappa_{(3)})$$
$$d \mathrel{\widehat{=}} H(s_{(1)}, s_{(2)}, s_{(3)}, s_{(4)}) = H(\kappa_{(1)}, \kappa_{(2)}, \kappa_{(3)})$$

The forms a and b can be classified as imaginary condensations, the forms c and d as complex ones, as here also the real unit $s_{(1)}$ becomes hermetric. The form a denotes condensations in the trans-coordinates x_5, x_6, but b time condensations, c space condensations and d spacetime condensations.

Here we have also introduced a new quantity $\kappa_{(\mu)}$ which denotes the single distinguishable structure units – Heim calls them lattice kernels – $\kappa_{(1)} = \kappa_{(1)}(x_5, x_6)$, $\kappa_{(2)} = \kappa_{(2)}(x_4)$, $\kappa_{(3)} = \kappa_{(3)}(x_1, x_2, x_3)$ (again 3 units, compare above) with the properties $\kappa_{(\mu)} \neq \kappa^{\times}_{(\mu)}$ and $g_{(\mu\nu)} = \mathrm{Tr}(\kappa_{(\mu)} \times \kappa_{(\nu)})$, the $g_{(\mu\nu)}$ being the partial structure fields which make up the composition field g_x per hermetry as given in Appendix E.

3.4.4 General solution of the hermetric fundamental problem

Prior to an analysis of the single hermetry forms we want to find a solution for the general hermetric problem. This means that Eq. (28), which we now write in the component form using C_p from (19)

$$\partial_l \left\{ {i \atop km} \right\} - \partial_m \left\{ {i \atop kl} \right\} + \left\{ {i \atop ls} \right\}\left\{ {s \atop km} \right\} - \left\{ {i \atop ms} \right\}\left\{ {s \atop kl} \right\} = \lambda_m(k,l) \left\{ {i \atop kl} \right\}, \qquad (30)$$

has to be solved for the general situation of $1 < q \le 6$ hermetric coordinates in the R_6. If the covariant components k and l, respectively, k or l, are anti-hermetric, then $\lambda_m(k,l) = 0$ follows due to the definition of anti-hermetry. If in contrast k and l run in the domain of the q hermetric coordinates and if $m = \tilde{m}$ is anti-hermetric, then due to $\left\{ {i \atop \tilde{m}l} \right\} = 0$ and $\left\{ {i \atop kl} \right\} = \mathrm{const}(Z^{\tilde{m}})$ always $\lambda_{\tilde{m}}(k,l) \left\{ {i \atop kl} \right\} = 0$ follows which, due to $\left\{ {i \atop kl} \right\} \neq 0$, can only be fulfilled by $\lambda_{\tilde{m}}(k,l) = 0$. This means that the eigenvalue spectra of covariant hermetric components of the condenser cannot have components in the anti-hermetric structure units of the world. By the existence of such anti-hermetric units these eigenvalue spectra remain empty: $\lambda_{\tilde{m}}(k,l) = \lambda_m(\tilde{k},l) = \lambda_m(k,\tilde{l}) = 0$. Beside these identities there are relations between non-empty spectra: Writing Eq. (30) for $k = m$ (Eq. (1)) and for exchanged indices m and l and $k = m$ (Eq. (2)), then the sum of (1) and (2) provides the symmetry relation $\lambda_m(m,l) \left\{ {i \atop ml} \right\} + \lambda_l(m,m) \left\{ {i \atop mm} \right\} = 0$. Defining the relation $a_{ml} = -\frac{\lambda_l(m,m)}{\lambda_m(m,l)}$, we can write $\left\{ {i \atop ml} \right\} = a_{ml} \left\{ {i \atop mm} \right\}$ and similar $\left\{ {i \atop lm} \right\} = a_{lm} \left\{ {i \atop ll} \right\}$. Based on these relations and on $\widehat{\{\}} = \widehat{\{\}}^{\times}$ being hermitian, one can transform Eq. (30) to

$$\left((a(k,l) - 1)\, \partial_l - \sum_{m \neq l} \partial_m \right) \phi_{kl} + \phi_{kl}^2 = \lambda(k,l)\phi_{kl} \qquad (31)$$

with the abbreviations and quantities $a(k,l), b_i(k,l), \lambda(k,l)$ and the function $\phi_{kl} = b_i(k,l) \left\{ {i \atop kl} \right\}$ as defined and derived in Appendix F. This is a Bernoulli differential equation which can be solved in a straightforward way, see Appendix F, leading to a result expressed through the normalised function

$$\psi_{kl} = \frac{\phi_{kl}}{\lambda(k,l)} = \frac{b_i(k,l)}{\lambda(k,l)} \left\{ {i \atop kl} \right\} = \left(1 + C_{kl}\, e^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1} = \left(1 + C_{kl}\, e^{-\lambda_{kl} x}\right)^{-1} \qquad (32)$$

where $\vec{\lambda}_{kl}, \lambda_{kl}$ are functions of the $\lambda(k,l), a(k,l)$ and (in case of λ_{kl}) the angles of the q coordinates as defined in Appendix F, C_{kl} a constant of integration and x the q-dimensional "length" with $x^2 = \sum_{i=1}^{q} x_i^2$. From this solution also the fundamental metric tensor g_{ik} can be calculated as given in Appendix F. The function ψ_{kl} has the extrema $\psi_{kl}^{(\min)} = 0$ and $\psi_{kl}^{(\max)} = 1$, which turns out are the extrema of the g_{ik} as well. The maximum of 1 is reached for $(\lambda_{kl}x)_{ext} = \alpha + i\beta$ if $e^{-\alpha}e^{-i\beta} = e^{-\alpha}(\cos\beta - i\sin\beta) = 0$, which is only achievable for $\cos\beta - i\sin\beta = 0$ or for $\cos\beta = 0$ and $\sin\beta = 0$ when $\alpha < \infty$. So, the imaginary part $\mathrm{Im}(\lambda_{kl}x)$ provides in principle four discrete spectra $\beta_+^{(\pm)} = \pm\frac{\pi}{2}(2n_+ + 1)$ from $\cos\beta_+ = 0$ and $\beta_-^{(\pm)} = \pm\pi n_-$ from $\sin\beta_- = 0$. The real part $\mathrm{Re}(\lambda_{kl}x)$ remains instead without effect concerning the formation of discrete eigenvalues.[20] The existence of the three imaginary world coordinates obviously causes the hermetric condensation (metric curvature) and so quantum levels in the R_3 projection.

We now consider the dependency on x. As the real coordinates of the $R_3 \widehat{=} s_{(1)}$ always appear only as one structure unit, we can set for these real quantities $r^2 = \sum_{i=1}^{3} x_i^2$ so that the remaining $q-3$ imaginary coordinates can be combined to $-\xi^2 = \sum_{s=4}^{q} x_s^2$. Then we get $\pm ix = \sqrt{\xi^2 - r^2}$ and with $y = ix$

$$\psi_{kl} = \left(1 + C_{kl}\, e^{\pm i\lambda_{kl} y}\right)^{-1}, \quad y^2 = \xi^2 - r^2. \tag{33}$$

As long as $r < \xi$ holds, due to $e^{\pm i\lambda_{kl}y} = \cos(\lambda_{kl}y) \pm i\sin(\lambda_{kl}y)$ discrete eigenvalue spectra of quantum levels occur. Only for $r \geq \xi$, thus $y = i\sqrt{r^2 - \xi^2}$ the exponent becomes real and ψ_{kl} exponentially approaches a constant limit. This indicates the existence of the fourth area of validity, the macroscopic area. If $r \gg \xi$, then $y \approx ir$ and the quantum spectra become so dense that the eigenvalues $\lambda_{kl} \to \lambda$ approach a continuum in which the single elements (k, l) are no longer distinguishable, so that $\psi_{kl} \to \psi$ and $C_{kl} = C$. In this macroscopic approximation

$$r^2 \gg \xi^2, \quad \psi = (1 + Ce^{\pm\lambda r})^{-1} \tag{34}$$

an exponential decay law exists for both branches.

3.4.5 Hermetric elementary structures

Now the hermetry forms a to d shall be analysed regarding their physical interpretation. The forms a and b can be classified as imaginary condensations, since the real coordinates of the R_3 do not occur in them. They are not the ones determining the masses of the elementary particles, which are the main subject of our consideration. Therefore, we present Heim's findings for the forms a and b only in a short summary:

As the hermetry form a describes terms in x_5 and x_6 outside the R_4, a physical interpretation seems to be impossible. It is only conceivable that the structure a has an indirect impact on the R_4. This is the case if a fulfils the condition of a null geodesic, because then the R_4-sector of the fundamental metric tensor is pseudo-Euclidian (due to its anti-hermetry) and the metric quantities of a, for $ds^2 = 0$, depend on the R_4-coordinates. So, even if a is not directly physically explainable, its effect on the (anti-hermetric) R_4 can be analysed, see Appendix G for details. According to Heim [43] it turns out that it obviously causes a gravitational disturbance which spreads out with a finite velocity. As the eigenvalues build discrete spectra in the a hermetry, too, these advancing gravitational fields must have the character of discrete quantum levels as well, which therefore are identified as gravitons by Heim. The fact that x_5 and x_6 appear in all hermetry forms as hermetric coordinates (as lattice kernel $\kappa_{(1)}$, compare Subsection 3.4.3) explains, so Heim, why gravity is always present for all matter.[21]

In form b, all three imaginary coordinates x_4, x_5 and x_6 are hermetric and this means that an analogous formalism as for the form a exists. The obtained relation can be extended into the R_4, where with the anti-hermetric R_3 coordinates geodesic null lines exist $ds^2 = dr^2 + dx_4^2 = dr^2 - c^2t^2 = 0$ on the light cone. The R_3 coordinates then occur together with the time coordinate as well-known Lorentz invariant term in the equation $(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})P = \frac{\lambda^2}{4}P$ with P as state function, which is for $\lambda \approx 0$ identical with a transversal electromagnetic wave field. Therefore, Heim concludes that the imaginary time condensations of the Hermetry form b describe photons, the imponderable quanta of the electromagnetic field [43].

The results of Heim's mathematical analysis of forms a and b can be found in Appendix G (but can be skipped in first reading as not needed for the further outline).

The Hermetry forms c and d have in common that they both contain the semantic unit $s_{(1)}$ with the real coordinates x_1, x_2, x_3 beside imaginary coordinates, so that the multi-dimensional world vector and the line element become complex $\vec{x} = \vec{r} + i\vec{\xi}$, $x = \alpha + i\beta$. As in the case of form a

20 As Heim, we use the term 'eigenvalue' for these discrete spectra, defined by the maxima of function ψ_{kl}, although they are not obtained by the non-linear eigenvalue equation (30), respectively (31) alone, which first still allow a continuous solution set. Other than, e.g. in case of the Schrödinger equation for bound states in quantum mechanics (see for instance [60]), here the requirement of a normalised (wave) function cannot be used as a constraint to restrict the solutions to discrete values. Heim's approach considering the maxima therefore appears plausible.

21 It should be emphasised that Heim's argument as given in Appendix G may support this explanation and give a hint for gravitational field quanta, as stated. But the gravitational field on the macroscopic level of course is described by the transition to this macroscopic realm and so by the Einstein equations of GR which hold in Heim's theory in this realm, as already explained.

(see Appendix G), the coefficient C_{kl} turns out to be -1 and $\psi_{kl} = \left(1 - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1} = \left(1 - \mathrm{e}^{-\lambda_{kl}x}\right)^{-1} = \left(1 - \mathrm{e}^{\pm i\lambda_{kl}y}\right)^{-1}$ with $y^2 = \xi^2 - r^2$, $\xi_c^2 = \epsilon^2 + \eta^2$, $\xi_d^2 = \epsilon^2 + \eta^2 + c^2t^2$. A short analysis of possible singularities shows that in the third area of validity the condition $\mathrm{e}^{-\vec{\lambda}_{kl}\vec{r}}(\cos\vec{\lambda}_{kl}\vec{\xi} - \mathrm{i}\sin\vec{\lambda}_{kl}\vec{\xi}) = 1$ causes a pole in ψ_{kl}, which means that only for $\sin\vec{\lambda}_{kl}\vec{\xi} = 0$, i.e. $\vec{\lambda}_{kl}\vec{\xi} = n\pi$ and $\cos\vec{\lambda}_{kl}\vec{\xi} = \mathrm{e}^{\vec{\lambda}_{kl}\vec{r}}$ a singularity occurs. The last relation can only be fulfilled if $\vec{\lambda}_{kl}\vec{r} \le 0$. For $r \to 0$ the even-numbered eigenvalues $\cos\vec{\lambda}_{kl}\vec{\xi} = +1$, i.e. $\vec{\lambda}_{kl}\vec{\xi} = 2n\pi$ lead to such a singularity which has to be excluded as unphysical. So, only the imaginary values $\sin\vec{\lambda}_{kl}\vec{\xi} = \pm 1$ and the odd-numbered real values $\cos\vec{\lambda}_{kl}\vec{\xi} = -1$ count without restriction, the even-numbered real values only if $r > 0$.

Together with the findings for the maxima of ψ_{kl} in Subsection 3.4.4 this means that we end up with the physically allowed spectra $\vec{\lambda}^+_{kl}\vec{\xi} = \left(\vec{\lambda}^+_{kl}\vec{e}_\xi\right)\xi = \lambda^+_{kl\,\xi}\xi = \pm\frac{\pi}{2}(2n_+ + 1)$ and $\lambda^-_{kl\,\xi}\xi = \pm\pi(2n_- + 1)$, which includes $\lambda^-_{kl\,\xi}\xi = 2\lambda^+_{kl\,\xi}\xi$, and with $\lambda^-_{kl\,\xi}\xi = \pm 2\pi n_-$ for $r > 0$. It is now possible to derive a relation between ξ and r for these spectra, when noticing that similar relations as for $\lambda^\pm_{kl\,\xi}\xi$ can be inserted for $\lambda^\pm_{kl\,\xi}y$ and $\lambda^\pm_{kl\,\xi}r$ if a certain mathematical relation is respected:

$$\left(\lambda^-_{kl\,\xi}\right)^2\xi^2 = \left(\lambda^-_{kl\,\xi}\right)^2(r^2 + y^2) = \pi^2((2n_r + 1)^2 + (2n_y)^2)$$
$$= \pi^2(2n_- + 1)^2 \tag{35}$$

For the $\lambda^-_{kl\,\xi}\xi = \pm 2\pi n_-$ Eq. (35) changes to a relation with even $2n_r$ and $2n_-$. The equation holds for the integers n_-, n_y, n_r if the squared terms on the r.h.s. form a Pythagorean triple.[22] Only then the n_- is a natural number and fulfils the eigenvalue condition above. After a term by term equalisation relation (35) can be transformed to $\left(\lambda^-_{kl\,\xi}\right)^2\xi^2 = \pi^2(2n_- + 1)^2$ and $\left(\lambda^-_{kl\,\xi}\right)^2 r^2 = \pi^2(2n_r + 1)^2$ and by division to

$$\frac{\xi}{r} = \frac{2n_- + 1}{2n_r + 1}, \quad = 1 \text{ if } n_r = n_-,$$
$$\text{or} \quad = \frac{2n_- + 1}{2n_- - 1} \text{ for } n_r = n_- - 1, \ \to \ 1 \text{ for } n_- \gg 1. \tag{36}$$

22 In Eq. (35) with odd terms for ξ (r.h.s.) a necessary condition is met by having a combination of an odd and even number on the l.h.s. Alternative to (35) this condition can also be fulfilled by $\pi^2((2n_y + 1)^2 + (2n_r)^2) = \pi^2(2n_- + 1)^2$. The three terms which are squared in (35) can be generated by arbitrary integers $u, v \in N, u > v$ via $2n_- + 1 = u^2 + v^2$, $2n_y = 2uv$, $2n_r + 1 = u^2 - v^2$. In the 'even' case, analogous relations apply to the $2n_-$, $2n_y$ and $2n_r$.

Note that $\xi \ne r$, i.e. $y > 0$ only has to be requested if $r = 0$ to exclude the above mentioned singularity. It can easily be seen that the same result is obtained for the spectrum $\lambda^+_{kl\,\xi}\xi = \pm\frac{\pi}{2}(2n_+ + 1)$ and that an equation $\frac{\xi}{r} = \frac{n_-}{n_r}$ follows for the $\lambda^-_{kl\,\xi}\xi = \pm 2\pi n_-$ if $r > 0$.

We now want to consider the time dependency of $\xi(t)$ and $r(t)$ (according to [43]). In general, independent of the complex Hermetry form, ξ must be a time dependent function, as otherwise no interaction could exist for the respective condensation, which could only be fulfilled if there were no other condensation in the R_6. But this would contradict experienced reality. This means $\mathrm{d}y^2 = \mathrm{d}\xi^2 - \mathrm{d}r^2 = (\dot{\xi}^2 - \dot{r}^2)\mathrm{d}t^2 = w^2(1 - v^2/w^2)\mathrm{d}t^2$, as $\dot{\xi} = w$ is the imaginary part of the world velocity in Euclidian or pseudo-Euclidian approximation and $\dot{r} = v$ a temporal change of the position in the R_3. So, with $\beta = v/w$ we get $y = \int w\mathrm{d}t\sqrt{1-\beta^2} = \xi\sqrt{1-\beta^2} + \int \xi\beta\dot{\beta}(1-\beta^2)^{-1/2}\mathrm{d}t$. As in general $\lambda_{kl}y \ne 0$ $(r > 0)$ and $\lambda_{kl} \ne 0$, also $y \ne 0$ holds. If further $\beta = \text{const}$, which always must be possible, then $\dot{\beta} = 0$ and $y = \xi\sqrt{1-\beta^2} \ne 0$, which demands $\beta^2 \ne 1$. Finally the algebraic character of y may not change through the request $\dot{\beta} = 0$, so that the constraint $1 - \beta^2 \ne 0$ is stated more precisely as $1 - \beta^2 > 0$, i.e. $0 \le \beta < 1$. This characterises the pseudo-Euclidian R_6 in which a null geodesic is not reached. This in turn is characteristic for massive ponderable quanta, so that we can conclude that in case of the forms c and d the λ_{kl} describe quantum levels of massive particles. The physical difference between the forms c and d becomes clear by realising that these complex-valued forms can be understood as couplings of the imaginary forms a and b to the metric R_3 structure. And since form b is to be interpreted as describing the photon, the form d must be considered as a condensation which in some way is coupled with a photonic, i.e. electromagnetic field, in fact as a consequence of the time dimension involved in the condensation process. So, the Hermetry form d obviously describes charged particles and the form c neutral.

An open question is, whether and how the strong and the weak interaction with their gauge symmetries could be derived from the hermetry forms. We consider the possible relation between the 6-dimensional Heim space and these gauge symmetries in Appendix C, subsection C.3, and the r-dependent course of the interaction, in this case obviously the strong interaction, as it must dominate the condensation and so generation of mass, in Appendix L. We find that this course seems to have a linear increase in the area between an inner core and the outer asymptotic region, which suggests an analogy to the confinement potential of the QCD.

3.5 Poly-metric geometry and partial solutions

3.5.1 Basics

After these preliminary investigations we come to a central aspect of Heim's theory, which was already heuristically introduced in Subsection 3.4.2, the existence of partial metric structures and the composition of the overall hermitian geometry in the R_6 of such structures, leading to a "poly-metric" geometry. According to Heim, this poly-metric allows for a refinement of the Riemannian geometry so that internal structures (processes) of matter become describable on a geometric basis. The fundamental metric tensor g_{ik} therefore is composed of a combination of partial structures as suggested by the different semantic geometric structures and groups of coordinates $s_{(j)}$ already introduced in Subsection 3.4.3.

Starting with the metric tensor in an Euclidian or pseudo-Euclidian coordinate system (assumed to be orthogonal) η_i, we can derive its form for a system of non-Euclidian coordinates x_i, but also can assume the existence of another coordinate system y_i which stands for the partial structures $\kappa^{(\mu)}$:[23]

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{\partial \eta_m}{\partial x^i}\frac{\partial \eta_m}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k \\
&= \sum_{\alpha,\beta=1}^{6}\left(\frac{\partial \eta_m}{\partial y^\alpha}\frac{\partial y_\alpha}{\partial x^i}\right)\left(\frac{\partial \eta_m}{\partial y^\beta}\frac{\partial y_\beta}{\partial x^k}\right)\mathrm{d}x^i\mathrm{d}x^k \\
&= \sum_{\alpha,\beta=1}^{6}\kappa^{(\alpha)}_{im}\kappa^{(\beta)}_{mk}\mathrm{d}x^i\mathrm{d}x^k \\
&= g_{ik}\mathrm{d}x^i\mathrm{d}x^k
\end{aligned}
\tag{37}
$$

We now can structure this expression according to the coordinate groups $s_{(j)}$ where we join $s_{(3)}(x_5)$ and $s_{(4)}(x_6)$ to one group as they appear only as one unit (as already seen in the analysis of the hermetry forms),

$$
\begin{aligned}
g_{ik} &= \sum_{\alpha=1}^{6}\kappa^{(\alpha)}_{im}\sum_{\beta=1}^{6}\kappa^{(\beta)}_{mk} \\
&= \left(\sum_{\alpha=1}^{3}\kappa^{(\alpha)}_{im}+\kappa^{(4)}_{im}+\sum_{\alpha=5}^{6}\kappa^{(\alpha)}_{im}\right) \\
&\quad\times\left(\sum_{\beta=1}^{3}\kappa^{(\beta)}_{mk}+\kappa^{(4)}_{mk}+\sum_{\beta=5}^{6}\kappa^{(\beta)}_{mk}\right) \\
&:= (\kappa^{(3)}_{im}+\kappa^{(2)}_{im}+\kappa^{(1)}_{im})(\kappa^{(3)}_{mk}+\kappa^{(2)}_{mk}+\kappa^{(1)}_{mk}) \\
&= \sum_{\mu=1}^{3}\kappa^{(\mu)}_{im}\sum_{\nu=1}^{3}\kappa^{(\nu)}_{mk} \\
&= \sum_{\mu,\nu=1}^{3} g^{(\mu\nu)}_{ik}
\end{aligned}
\tag{38}
$$

where α, β run over the coordinates and μ, ν over the three partial structures $\kappa^{(\mu)}$. The κ depend on the coordinates as already noted in Subsection 3.4.3. As we have learned in that subsection as well through the hermetry forms, there are four complexes of metric tensors which can be composed of the partial structures: $g^{(x)}_{ik} = g^{(x)}_{ik}(g^{(\mu\nu)}_{ik}, \kappa^{(\mu)}_{ik})$ with $x = a, b, c, d$; $\mu, \nu = 1, 2, 3$; $i, k = 1, \dots, 6$. The concrete form of the $g^{(x)}$, which correlates the partial metric structures, therefore also called correlators, is given in Appendix E. We now can define poly-metric state functions, i.e. fundamental condensers (in Heim's terminology) depending on the partial metric structures. We denote the composition state function like Heim as $\begin{bmatrix} i \\ kl \end{bmatrix}$:[24]

$$
\begin{aligned}
\begin{bmatrix} i \\ kl \end{bmatrix} &= g^{is}\begin{bmatrix} \\ skl \end{bmatrix} \\
&= \frac{1}{2}\sum_{\kappa,\lambda=1}^{3} g^{is}_{(\kappa\lambda)}\sum_{\mu,\nu=1}^{3}\left(\frac{\partial g^{(\mu\nu)}_{ls}}{\partial x^k}+\frac{\partial g^{(\mu\nu)}_{ks}}{\partial x^l}-\frac{\partial g^{(\mu\nu)}_{lk}}{\partial x^s}\right) \\
&:= \sum_{\kappa,\lambda,\mu,\nu=1}^{3}\begin{bmatrix} i \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} \\
&:= \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}}
\end{aligned}
\tag{39}
$$

So, in the abbreviated syntax the $(\kappa\lambda)$ denote the contravariant and the $(\mu\nu)$ the covariant basic signature. The notation also expresses that the contravariant index i is provided by the partial structure κ. The index s of the partial structure λ facilitates the summation. In the metron picture the condensers cause that metrons, lying on the curved metron lattice and this being projected on a flat plane, are compressed or condensed. They mediate external interactions. As we see from (39), due to the

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23 As always, if not explicitly defined differently, the usual sum convention over identical indices holds. In this compact introduction of the poly-metric geometry we follow the summary of [55].

24 Heim already denotes all state functions/condensers with the squared brackets [ ] (instead of { }) after having introduced his metron calculus, meaning that the quantities are based on it. We use the squared brackets only if we want to denote state functions/condensers based on partial metric structures.

three partial structure units, there are $3^4 = 81$ fundamental condensers in the poly-metric, instead of the Christoffel symbols in the Riemannian geometry, so a much richer overall structure. Other than in Riemannian geometry, in which the product of co- and contravariant metric tensor gives the Kronecker symbol $g^{ik}g_{kj} = \delta^i_j$, in the poly-metric theory this product in general is a function $f^i_j(\alpha) \neq \delta^i_j$ with $\alpha \hat{=} \binom{\kappa\lambda}{\mu\nu}$ which expresses the correlation between the elements $(\kappa\lambda)$ and $(\mu\nu)$. This means that $\begin{bmatrix} i \\ kl \end{bmatrix}$ becomes

$$\begin{bmatrix} i \\ kl \end{bmatrix} = \sum_{\kappa,\lambda,\mu,\nu=1}^{3} \left(\begin{bmatrix} i \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} + Q^i_m \begin{pmatrix} \kappa\lambda \\ \mu\nu \end{pmatrix} \begin{bmatrix} m \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} \right)$$
$$= \sum_{\kappa,\lambda,\mu,\nu=1}^{3} \left(\widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}} + \operatorname{Tr} Q \begin{pmatrix} \kappa\lambda \\ \mu\nu \end{pmatrix} \times \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}} \right) \quad (40)$$

with $Q\begin{pmatrix} \kappa\lambda \\ \mu\nu \end{pmatrix}$ being a so-called correlation tensor which couples the respective partial structures. If g_{ik} consists of only one partial structure μ, then no correlation exists, i.e. $Q\begin{pmatrix} \mu\mu \\ \mu\mu \end{pmatrix} = 0$, as in the monometric Riemannian geometry of GR.[25]

Starting from the fact that a condenser with an anti-hermetric covariant index $\tilde{k}$ always is zero, $\begin{bmatrix} i \\ \tilde{k}l \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} = 0$, for the metric tensor in the poly-metric the subsequent interesting symmetry properties can be derived, see Appendix H, with $\pm$ denoting the hermitian and the anti-hermitian parts of the tensors and V_r being the hermetric subspace of the R_6,

$$g^{(\mu\nu)}_{+\tilde{k}l} = \text{const}, \qquad g^{(\mu\nu)}_{-\tilde{k}l} = 0, \qquad g^{(\mu\nu)}_{+kl} = g^{(\mu\nu)}_{+kl}(V_r),$$
$$g^{(\mu\nu)}_{-kl} = \text{const} \neq 0, \quad (41)$$

from which follows that the poly-metric condensers are hermitian (Appendix H):

$$\widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}} = \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}}_+ = \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}}^\times \quad (42)$$

The anti-hermitian part of $g^{(\mu\nu)}_-$ turns out to be expressable through a spin field tensor $g^{(\mu\nu)}_{-kl} = P^{(\mu\nu)}_{kl} = (rot_{(x)}\vec{\phi}^{(\mu\nu)})_{kl}$ with the spin field vector $\vec{\phi}$. A non-zero spin evolves if the orientations of metrons do not cancel out. In the overall metric tensor of the composition field $g = g^\times$ no anti-hermitian part exists and therefore no spin tensor either, where in contrast in the partial structures and correlators spin fields always appear, presupposed that there is at least one structure unit in which metric condensations exist, i.e. which is hermetric.

3.5.2 Solution of the fundamental poly-metric problem

With the definition of the tensor $F^{(\kappa\lambda)i}_{(\mu\nu)kl} = \begin{bmatrix} i \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} + Q^i_m \begin{pmatrix} \kappa\lambda \\ \mu\nu \end{pmatrix} \begin{bmatrix} m \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)}$ an insertion of expression (40) into Eq. (30) gives

$$\partial_l F^i_{km} - \partial_m F^i_{kl} + F^i_{sl} \begin{bmatrix} s \\ km \end{bmatrix} - F^i_{sm} \begin{bmatrix} s \\ kl \end{bmatrix} = \underline{\lambda}_m(k,l) F^i_{kl} \quad (43)$$

after a term by term equation under the sum over the indices $\kappa, \lambda, \mu, \nu$ (here suppressed) for the general poly-metric situation, with $\begin{bmatrix} i \\ kl \end{bmatrix}$ as defined in (40), being the composition field.[26] From this form immediately the same relations for the $\underline{\lambda}_m(k,l)$ follow as from Eq. (30) for the compositive case if one or more indices are anti-hermetric. With these relations and similar steps as in the solution of the composition field (Appendix F) we find the solution

$$F^{(\kappa\lambda)i}_{(\mu\nu)kl} = A^{(\kappa\lambda)i}_{(\mu\nu)kl} \exp\left(\vec{\underline{\lambda}}_{kl}\vec{x} - \int \sum_{j=1}^{q} \mathrm{d}x_j \underline{C}_{js} \begin{bmatrix} s \\ kl \end{bmatrix} \right) \quad (44)$$

with A^i_{kl} as integration constant and the other coefficients as defined in Appendix H, see there (135) and previous equations for the details. The underlining of coefficients means that these quantities belong to the special poly-metric solution and therefore depend in general on the (for simplicity of the notation suppressed) partial structure indices, in contrast to the not underlined coefficients of the composition field. The result (44) means that the problem of solving the fundamental equation of the poly-metric can be reduced to an integral over the condenser (state function) of the composition field. We therefore insert this result, given in equation (32), in (44) and obtain the final expression

$$F^{(\kappa\lambda)i}_{(\mu\nu)kl} = C^{(\kappa\lambda)i}_{(\mu\nu)kl}\, \mathrm{e}^{\vec{\underline{\lambda}}_{kl}\vec{x}} \left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1 \right)^{-\underline{\alpha}_{kl}} \quad (45)$$

with the single steps, constants and coefficients given in subsection H.2 of Appendix H. The further analysis of the result (in that appendix) provides that under a certain condition (see for reference (140)) the relation $\vec{\underline{\lambda}}_{kl} = \underline{\alpha}_{kl}\vec{\lambda}_{kl}$

25 Here we should emphasize that the poly-metric still contains two partial structures in the R_4, $\kappa^{(2)}$ and $\kappa^{(3)}$. But in the macroscopic realm of GR the r.h.s. of Eq. (17) becomes the macroscopic energy momenum tensor as shown in (18), which "forces" the l.h.s., i.e. the geometric side in (17) to behave as one metric structure.

26 Heim calls the tensor F (in German) "Fremdfeldselektor" which possibly could be translated with "Extrinsic field selector", but does not seem very illuminating.

holds and then the poly-metric state function can be written in a simpler form as

$$\underline{\psi}_{kl} = e^{\vec{\underline{\lambda}}_{kl}\vec{x}} \left(e^{\vec{\lambda}_{kl}\vec{x}} - 1\right)^{-\underline{\alpha}_{kl}} \rightarrow \left(1 - e^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-\underline{\alpha}_{kl}} = \psi_{kl}^{\underline{\alpha}_{kl}} \tag{46}$$

with $\underline{\alpha}_{kl}$ being an exponent which contains the strength of the correlations, i.e. interactions within the poly-metric. In the limit, when the respective parameters which determine $\underline{\alpha}_{kl}$ become equal to those of the composition field, $\underline{\alpha}_{kl}$ becomes 1 so that $\underline{\psi}_{kl} \rightarrow \psi_{kl}$, i.e. the field becomes identical to the composition field, as to be expected. So, a quantitative solution of the poly-metric fundamental problem results in the calculation of the correlation exponent $\underline{\alpha}_{kl}$ (and the $\vec{\underline{\lambda}}_{kl}$ in the general case). Note that $\underline{\alpha}_{kl}$ can become positive or negative, depending on the mentioned parameters (and these on the indices k, l). A negative value does not automatically lead to an infinity of F^i_{kl}, since its course also depends on the quantity $\vec{\underline{\lambda}}_{kl}\vec{x}$, which in turn can be related to $\underline{\alpha}_{kl}$ (see (144) in Appendix H) and in general can have a diverse behaviour in the coordinate space.

The extremum condition $\partial F^i_{kl} = 0$ analysed in Appendix H does not only lead to a relation between the $\underline{\lambda}(k,l)$ and $\lambda(k,l)$ (see the appendix), but can also be met by $F^i_{kl} = 0$ if the exponent in $\exp(\vec{\underline{\lambda}}_{kl}\vec{x})$ is complex $\vec{\underline{\lambda}}_{kl}\vec{x} = \underline{a} + i\underline{\beta}$ so that $e^{\underline{a}}(\cos\underline{\beta} + i\sin\underline{\beta}) = 0$ is to be fulfilled by $\cos\underline{\beta} = 0$ or $\sin\underline{\beta} = 0$ for the real or imaginary part. This leads to the same eigenvalue spectra as in the case of the composition field: $\underline{\beta}^{(\pm)}_{+} = \pm\frac{\pi}{2}(2\underline{n}_{+} + 1)$, $\underline{\beta}^{(\pm)}_{-} = \pm\pi\underline{n}_{-}$. Furthermore, the relations $(\vec{\lambda}_{kl}\vec{r})_{\text{ext}} = d \cdot (\vec{\lambda}_{kl}\vec{\xi})_{\text{ext}}$ and $(\vec{\underline{\lambda}}_{kl}\vec{r})_{\text{ext}} = d \cdot (\vec{\underline{\lambda}}_{kl}\vec{\xi})_{\text{ext}}$ (not mentioned by Heim) with a parameter or function d can be derived at the extrema, and the $\underline{\beta}^{(\pm)}_{\pm}$ and $\beta^{(\pm)}_{\pm}$ of the composition field can be calculated as $\underline{\beta}^{(\pm)}_{\pm}/\beta^{(\pm)}_{\pm} = \underline{\alpha}_{kl} f_{\text{ext}}(k,l)$, see both in Appendix H, subsection H.2. Here the in general $\vec{x}$-dependent function $f(k,l)$ is introduced, $f_{\text{ext}}(k,l)$ being its value(s) at the extrema (eigenvalues), which does not appear in Heim's calculation (his calculation would mean $f(k,l) = 1$), but is obviously necessary to avoid a too rough approximation, see again Appendix H, subsection H.2. We shall see below and in Section 4 that $f(k,l) \neq 1$ plays an important role in our further calculations, as together with the eigenvalues it determines the exponent $\underline{\alpha}_{kl}$:

$$\underline{\alpha}_{kl} = \frac{\underline{\beta}^{(\pm)}_{\pm kl}}{\beta^{(\pm)}_{\pm kl}} f^{-1}_{\text{ext}}(k,l) \tag{47}$$

Here we explicitly marked the dependency on the coordinate indices.

3.5.3 Correlations and classes of condensation

Based on the results (45) and (46) Heim [44] derives expressions for the metric tensor and the correlation tensor, see Appendix H, subsection H.3. From these expressions the roots, extrema and also second derivatives, thus inflexion points can be analysed. It turns out that the extrema of the correlation tensor Q fall together with those of the condenser field ψ, and the second derivatives of the metric tensor result in the following form

$$\partial Q^{(\kappa\lambda)}_{(\mu\nu)} = 0, \quad \partial\psi_{ll} = 0, \quad \partial g_{(\kappa\lambda)} = 0, \quad \partial^2 g_{(\mu\nu)} = 0. \tag{48}$$

This can be interpreted this way (Heim [44]): The coupling extrema of all condensers (out of g) lie in the diagonal condensation levels of the composition field (ψ_{ll}). In this coupling sector the metric tensor components are extremal in both signatures of the condenser, in the basic covariant signature as inflexion point, thus here a pseudo-anti-hermetry seems to exist so that the number of (from g) buildable condensers in each coupling extremum is reduced. A variation of the couplings must lead to a restructuring of the composition field. Each coupling extremum $Q^{(\kappa\lambda)}_{(\mu\nu)\text{ext}} := \binom{\kappa\lambda}{\mu\nu}$ determines a coupling group containing all coupling extrema which arise from permutations of the signature figures. Heim performs a detailed analysis of all possible combinations and sorts them, depending on the associated hermetry forms and on the partial structure units. We only consider the a hermetry as simplest example: In this case there are only two correlating tensors, $g_{(11)} = \text{Tr}(\kappa_{(1)} \times \kappa_{(1)})$ and $\kappa_{(1)}$. They allow the construction of four condensers, namely $\begin{bmatrix}\kappa\kappa\\ \kappa\kappa\end{bmatrix}_+ := \begin{bmatrix}\\ \kappa\end{bmatrix}_+$, $\begin{bmatrix}g\\ g\end{bmatrix}_+ := \begin{bmatrix}\\ g\end{bmatrix}_+$ as well as $\begin{bmatrix}\kappa\\ g\end{bmatrix}_+$ and $\begin{bmatrix}g\\ \kappa\end{bmatrix}_+$, where co- and contravariant signature differ. As a correlation tensor couples different elements of g, always $Q^{(\alpha\alpha)}_{(\alpha\alpha)} = 0$ must hold for couplings of condensers with identical signatures. Thus, in the a hermetry a non-zero coupling, i.e. correlation tensors can only exist for the condensers $\begin{bmatrix}g\\ \kappa\end{bmatrix}_+$ and $\begin{bmatrix}\kappa\\ g\end{bmatrix}_+$. For the b and the c hermetry each 30 condensers arise, for the d hermetry $9^2 - 9 = 72$ condensers, as here all three partial structure units are involved.

Another insight can be gained from the geodesic law $\ddot{x}^i + \begin{bmatrix}i\\ kl\end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)} \dot{x}^k\dot{x}^l = 0$ which must hold for each condenser in the poly-metric. Indeed, for each element $(\mu\nu)$ of g a geodesic reference system can be found in which $\ddot{x}^i = 0$, but this property does not hold for each other element of g. So, if g consists of more than one structure unit, which is true for the hermetry forms b, c and d, no reference system can be found in which all condensations can be transformed away (as already stated in Section 3.4.2).

This is only possible for the condensations of form a in $g_{(11)}$, containing only the structure unit (1). In such a system $C_{(11)}$ all physical actions resulting from $g_{(11)}$ do no longer exist. As these obviously are gravitational actions (compare Subsection 3.4.5), gravitation appears as a reference-system-dependent "pseudo" force. Therefore the free fall is forceless, although the source of the gravitational field remains unchanged. In addition to this geodesic principle the principle of energy conservation, expressed through the relations (17) and (18), holds and leads to the basic equation of the composition field (28), meaning that the overall state of condensations expressed in this composition field is always conserved, independent of changes of the partial structures. In Heim's further analysis [44] it turns out that maxima of the coupling of partial structures ($\partial Q^{(\kappa\lambda)}_{(\mu\nu)} = 0$) fall together with minima of condensations (ψ) and vice versa. Combined with the conservation law of the composition field this means that a perpetual exchange mechanism between maxima and minima is possible, which Heim calls condensation flux.

We now focus on those sorts of condensations which obviously generate mass, i.e., as already derived on a principle level in Subsection 3.4.5 and above, on the condensers of the hermetry forms c and d which contain the partial metric structure $\kappa_{(3)}$ of the R_3. Only if this structure is involved, complex-valued condensers including a real part appear and cause condensations of the R_3. A real-valued condensation of the space can be used as a requirement and definiton to calculate the concrete state function $F_{(3)}$ of a pure spatial codensation

$$\mathrm{Im}F_{(3)} = \mathrm{Im}\left(\widehat{\begin{bmatrix}33\\33\end{bmatrix}}_+ + \mathrm{Tr}\left(Q^{(33)}_{(33)} \times \widehat{\begin{bmatrix}33\\33\end{bmatrix}}_+\right)\right) = 0 \quad (49)$$

where the results (44) and (45) can be inserted in the course of the calculation. Note that in this case, the composition field nonetheless may contain further hermetric coordinates, see the explanations below (54). Using the fact that the $\underline{\vec{\lambda}}_{kl}$ and the coefficients $\underline{c}_{is}$, composed of real eigenvalues, are real quantities and that the integral in Eq. (44) runs only over the three spatial real coordinates in this case, we get the result

$$F^i_{kl(3)} = \alpha^i_{kl}\, \mathrm{e}^{\underline{\vec{\lambda}}_{kl}\vec{r}} \left[\frac{1}{2}(1+\cos K) \right.$$
$$\left.(1+\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{r}} - \cos K\right)^2 \sin^{-2} K)\right]^{-\frac{\underline{\alpha}_{(3)}}{2}} \quad (50)$$

with the constant α^i_{kl}, $\vec{r}$ being the 3-dimensional position vector, $K = \vec{\lambda}_{kl}\vec{\xi}$ and $\vec{\xi} = \sum^q_{k>3} x_k \vec{e}_k$ the vector of the remaining hermetric coordinates of the R_6. See subsection H.4 in Appendix H for the detailed calculation and the expression of the exponent $\underline{\alpha}_{(3)}$ (which depends on the indices k, l, here suppressed).

From the result (50) immediately can be seen that the imaginary coordinates $k > 3$, which are present in all hermetry forms (at least with the coordinates x_5 and x_6), cause the spatial condensations, given through $F^i_{kl(3)} \neq 0$. Namely, if $K = 0$ (or equally $K = \pm 2\pi n$) then $\sin K = 0$ and $\cos K = 1$, and as $\underline{\alpha}_{(3)}$ will remain > 0 for a set of indices k, l (see Appendix H and subsection 4.2), $\lim_{K\to 0} F^i_{kl(3)} = 0$ holds for this area, i.e., a real-valued spatial condensation, which distinguishes the hermetry forms c and d, is only possible if $K \neq 0$. Next, it is crucial to analyse for which K extremal condensations occur. These obviously appear for the half-integral spectrum $K = \pm\frac{\pi}{2}(2n+1)$, as then $\cos K = 0$ and $\sin K = \pm 1$ hold. The following extremum of the condensation is then given

$$F^i_{kl(3)\,\mathrm{ext1}} = \alpha^i_{kl}\, \mathrm{e}^{\underline{\vec{\lambda}}_{kl}\vec{r}} \left(\frac{1}{2}\left(1+\mathrm{e}^{2\vec{\lambda}_{kl}\vec{r}}\right)\right)^{-\frac{\underline{\alpha}_{(3)}}{2}},\quad K = \pm\frac{\pi}{2}(2n+1). \quad (51)$$

In the realm of indices k, l where $\underline{\alpha}_{(3)} > 0$, this extremum with respect to K is a relative maximum, in the area of $\underline{\alpha}_{(3)} < 0$ a relative minimum (with $F^i_{kl(3)} \neq 0$). As we shall see in Section 4, the latter area obviously determines the energy-mass spectrum. So, as in variation principles, a spectrum starting with an energy minimum is found. But this half-integral spectrum is not the only set of extrema, depending on K. Also the odd integer spectrum $K = \pm\pi(2n+1)$ with $\cos K := b = -1$ and $\sin K = 0$ provides extrema, as then the terms $1 + \cos K = 1 + b$ and $\sin^2 K = 1 - b^2$ approach zero, which gives the following result for the square bracket in (50): $[\dots] = \frac{1}{2}(1+b) + \frac{1}{2}\frac{1+b}{1-b^2}\left(e^{\vec{\lambda}_{kl}\vec{r}} - b\right)^2$. The first summand becomes 0 and in the second we get $\frac{1+b}{1-b^2} = \frac{1}{1-b} \to \frac{1}{2}$ for $\cos K = b = -1$. Thereby we get the following result as second spectrum of extrema[27]

$$F^i_{kl(3)\,\mathrm{ext2}} = \alpha^i_{kl}\, \mathrm{e}^{\underline{\vec{\lambda}}_{kl}\vec{r}} \left(\frac{1}{4}\left(1+\mathrm{e}^{\vec{\lambda}_{kl}\vec{r}}\right)^2\right)^{-\frac{\underline{\alpha}_{(3)}}{2}},\quad K = \pm\pi(2n+1). \quad (52)$$

The results (51) and (52) provide important ingredients for our calculation of a concrete mass-energy spectrum in the R_3 in Section 4.

Before we concentrate on that, we briefly want to consider the course of $F^i_{kl(3)}$ in the R_3 and also analyse the further existing classes of condensation. The function

27 Heim does not mention this second spectrum of extrema of $F^i_{kl(3)}$, only the first one, which obviously is an incompletion.

$F^i_{kl(3)}$ becomes asymptotically for big r, i.e. $r \gg \xi$, thus with $\vec{\underline{\lambda}}_{kl}\vec{r} \approx \vec{\underline{\lambda}}_{kl}\vec{x} = \underline{\alpha}_{(3)} f(k,l)\vec{\lambda}_{kl}\vec{x} \approx \underline{\alpha}_{(3)} f(k,l)\vec{\lambda}_{kl}\vec{r}$ according to reference (144) and $b = \cos\vec{\lambda}_{kl}\vec{\xi}$:

$$F^i_{kl(3)} \sim e^{\vec{\underline{\lambda}}_{kl}\vec{r}}\left(\frac{1}{2}\frac{e^{2\vec{\lambda}_{kl}\vec{r}}}{1-b}\right)^{-\frac{\underline{\alpha}_{(3)}}{2}}$$
$$= (2(1-b))^{\frac{\underline{\alpha}_{(3)}}{2}} e^{(\vec{\underline{\lambda}}_{kl} - \underline{\alpha}_{(3)}\vec{\lambda}_{kl})\vec{r}}$$
$$= (2(1-b))^{\frac{\underline{\alpha}_{(3)}}{2}} e^{-\underline{\alpha}_{(3)}(1-f(k,l))\vec{\lambda}_{kl}\vec{r}} \quad (53)$$

In the here considered limit of $r \gg \xi$, $f(k,l)$ can become positive and > 1 (see Appendix H and subsection 4.2) and so $1 - f(k,l) < 0$. $\underline{\alpha}_{(3)}$ will be negative (as $\underline{\alpha}_{(3)} \sim f^{-1}_{\text{ext}}(k,l)$, see (47), and at least $f_{ext}(4,4) < 0$ in our parametrisation with fit to empirical data, see Subsection 4.2) so that the exponent in the result (53) in total becomes negative if $\vec{\lambda}_{kl}\vec{r} > 0$, and the function $F^i_{kl(3)}$ approximates zero for large r, as physically plausible. To allow for this behaviour puts a constraint on the $a(k,l)$ and $\underline{a}(k,l)$ which $f(k,l)$ depends on (see Subsection 4.2 with exemplary data).

Further *classes of condensation* can be derived from the following consideration: The result of reference (44) can be written as

$$F_{(\alpha\beta\gamma)}(x) = A^{i-1}_{kl}\left(\begin{bmatrix} i \\ kl \end{bmatrix} + Q^i_m \begin{bmatrix} m \\ kl \end{bmatrix}\right)$$
$$= \exp\left(\vec{\underline{\lambda}}_{kl}\vec{V} - \int \sum_{j=1}^{q_v} dV_j \underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix}\right) \quad (54)$$

where we have expressed the poly-metric indices by $\alpha\beta\gamma$, denoting the combination of the partial structures operating in F (e.g. $F_{(123)}$) and so the class of condensation. For simplicity, we have temporarily defined the F without the constant factor A^i_{kl}, x denotes the respective hermetry form and V the aggregate of the q_v hermetric coordinates of F. In general the condenser $[\ ] = \begin{bmatrix} i \\ kl \end{bmatrix} + Q^i_m \begin{bmatrix} m \\ kl \end{bmatrix}$ can have a different (i.e. lower) hermetry degree than the solution of the composition field $\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix}$. If $\vec{Q}$ is the q-dimensional coordinate aggregate of the composition field and $\vec{P}$ the remaining coordinate vector $\vec{P} = \vec{Q} - \vec{V}$, we get

$$F_{(\alpha\beta\gamma)}(x) = e^{\vec{\underline{\lambda}}_{kl}\vec{V}}\left(\frac{e^{\vec{\lambda}_{kl}\vec{V}} - e^{-\vec{\lambda}_{kl}\vec{P}}}{1 - e^{-\vec{\lambda}_{kl}\vec{P}}}\right)^{-\underline{\alpha}_v}$$
$$= \left(\frac{e^{\left(\vec{\lambda}_{kl} - \vec{\underline{\lambda}}_{kl}\underline{\alpha}_v^{-1}\right)\vec{V}} - e^{-\vec{\lambda}_{kl}\vec{P} - \vec{\underline{\lambda}}_{kl}\underline{\alpha}_v^{-1}\vec{V}}}{1 - e^{-\vec{\lambda}_{kl}\vec{P}}}\right)^{-\underline{\alpha}_v} \quad (55)$$

after a short calculation which is given in Appendix H, subsection H.5. The exponent $\underline{\alpha}_v$ has the same structure as in reference (136), but for the q_v hermetric coordinates, and becomes identical to $\underline{\alpha}$ if $\vec{V} = \vec{Q}$. (In this case, Eq. (45) holds for F.) The expression $\vec{\underline{\lambda}}_{kl}\underline{\alpha}_v^{-1}\vec{V}$ in the e-functions in (55) can be further evaluated by using again reference (144), obtaining $\vec{\underline{\lambda}}_{kl}\underline{\alpha}_v^{-1}\vec{V} = \vec{\lambda}_{kl}\vec{V} f_v(k,l)$ where $f_v(k,l)$ is the corresponding function $f(k,l)$ as in Appendix H, subsection H.2, but for the q_v hermetric coordinates of F, which then reads

$$F_{(\alpha\beta\gamma)}(x) = \left(\frac{e^{\vec{\lambda}_{kl}(1-f_v(k,l))\vec{V}} - e^{-\vec{\lambda}_{kl}(\vec{P}+f_v(k,l)\vec{V})}}{1 - e^{-\vec{\lambda}_{kl}\vec{P}}}\right)^{-\underline{\alpha}_v}. \quad (56)$$

The expressions which can be derived from (56) for the possible classes of condensation (according to the definition of the hermetry forms in Subsection 3.4.3) are listed in Appendix H, subsection H.5.

4 Derivation of a mass spectrum

4.1 Basic approach and calculation

We are now approaching the central objective of this paper which is to derive a formula for the mass spectrum of elementary particles from first principles. According to Heim this starts with the consideration that the r.h.s. of his fundamental equation (28), in the contracted form, represents an energy momentum density:

$$\text{Tr } C\{\} = \vec{\lambda}\{\} = \sum_i \lambda_i(k,l)\begin{bmatrix} i \\ kl \end{bmatrix} = \sum_{i,\alpha} \lambda_i(k,l)\begin{bmatrix} i \\ kl \end{bmatrix}_\alpha$$
$$= \sum_{i,\alpha} \lambda_i(k,l) F^i_{\alpha\, kl} := \sum_\alpha W^\alpha_{kl} \quad (57)$$

We have used the fact that the composition state function can be set as sum of partial structure state functions of the poly-metric (see (39)–(43)), and have summarised the set of indices of the partial structures in the index α. In general the W^α according to (57) will be complex, i.e. contain an imaginary part, which cannot correspond to a mass. So, the mass must be given by the real part Re(W), the imaginary part Im(W) instead obviously represents an energetic fluctuation width Γ and so, due to the quantum complementarity, the lifetime of the respective state (particle).

One obtains an energy (thus mass) by integrating the energy density over the 3-dimensional space. Therefore, as next step, the space volume element has to be defined in the frame of Heim's theory. The metronic character of space and thereby the quantised nature of energy, according to Section 3.3, has to be taken into account and with it the fact, that the metronic space element ∂V is not defined unambiguously, as $0 \leq s \leq 3$ coordinates can be metronic constants. Therefore, there must be 4 possible volume elements $\partial V_s = \beta_s \prod_{j=1}^{3-s} \partial_j \nu_j$ with $\beta_s = \text{const}$, ν_j being integer numbers (of metrons) and $\prod_{j=1}^{3-s} \nu_j = 1$ for $s = 3$.

Heim proceeds in his further approach by metronically integrating the fourfold volume element, obtaining a sum of 4 terms with different integer numbers and powers of them and elaborates the 'occupation numbers' of these numbers and the coefficients β_s by extensive heuristical and partially empirically related considerations, interconnected with his theory, which we do not want to explain here, see [44] for these details. On this path he achieves to define a set of so far complete quantum numbers, not only spin and isospin, but also a helicity and a number which corresponds to strangeness. He succeeds in obtaining a half-empirical mass formula which depends on the quantum numbers, but also on 36 fit parameters (which he artfully expresses through the set of mathematical constants π, e, ζ), and which is able to meet fairly well the empirical mass/energy data of the elementary particles, i.e. leptons, mesons and baryons and a wide area of their resonances which were known in the eighties (see [44] incl. listings of the particle masses which were calculated with Heim's formula at the DESY). Later (in 1989) Heim revised his mass formula, but did not publish this version. It was documented by the Research Group Heim's Theory in 2002.[28] This version does no longer depend on the mentioned fit parameters, but on different coefficients which are finally expressed through the set of quantum numbers. This seems like a big step forward, however, there is no derivation or explanation, how the new parts of the formula come about.

Nonetheless, in the eyes of the author, this work was a remarkable and creditable achievement and his formula in this way is still unique. But it has the shortcoming that it contains a lot of heuristic considerations which are not mandatory and by this do not directly follow from the equations of Heim's basic theory.

Therefore, we want to follow a different approach which foots directly on the equations and results described in this paper so far. We shall see that the calculation with a quantised space volume and discrete summation (instead of a continuous integration) will be crucial to achieve an expression which obviously can describe the phenomenology.[29] In our approach we shall not be able to derive a complete set of quantum numbers which correspond to those of modern particle physics, and thus, our results will not show a dependency of the energies/masses on such numbers, but only on two main quantum numbers which arise from our theory. But the results will show a clear correspondence to the phenomenological results presented and discussed in Section 2.

So, we continue[30] by calculating the mass/energy through evaluating the integral over the energy density

$$E = \int \sum_{s=0}^{3} \partial V_s \sum_{\alpha} W^{\alpha}_{k,l=4}$$
$$= \sum_{s=0}^{3} \beta_s \int \prod_{j=1}^{3-s} \partial_j \nu_j \sum_{\alpha,p} \lambda_p(k,l=4) F^{\,p}_{\alpha\,k,l=4}(\vec{x} = \vec{r} + i\vec{\xi}) \tag{58}$$

where the index p now denotes the contraction of the eigenvalues $\lambda_p(k,l=4)$ with the condenser state function of the partial structure $F^{\,p}_{\alpha\,k,l=4}(\vec{x})$ over the coordinates of the R_6. The further coordinate indices of the eigenvalues must be equal 4, denoting the x_4, i.e. the time coordinate, which marks the energy density within the energy momentum density tensor. In the following we suppress these fixed indices $k, l = 4$. As next step we realise that the integral with the quantised space volume element can be written as a multiple sum of discrete elements, each sum indexed with i_j $(1 \leq j \leq 3-s)$, and with $\Delta\nu_{i_j} = \nu_{i_j} - \nu_{i_j-1}$. Each product of the $\Delta\nu_{i_j}$ gives again an integer number n_{is}, which e.g. is $\Delta\nu^3$ for $s = 0$ and all $\Delta\nu_{i_j} = \Delta\nu$, and which can be a sum of such terms if the $F^{\,p}_{\alpha}(\vec{r}_{i_j}, \vec{\xi})$ fulfil, e.g. a spherical symmetry or are constant in a certain region (the larger the region, the bigger the n_{is}) so that the really different $F^{\,p}_{\alpha}(\vec{r}_i, \vec{\xi})$ can be grouped in the overall sum:

28 See document "Heim's Mass Formula (1989)" on http://heim-theory.com [55].

29 Note that a 'normal' volume integral over the energy density could not be calculated at all in a concrete way, since the integrand with the condenser state function depends on the special function f, which is not known in its full course, but only roughly qualitatively (see subsection 4.2 and Appendices H and L).

30 All further content and results were elaborated solely by the author.

$$E = \sum_{s=0}^{3} \beta_s \prod_{j=1}^{3-s} \sum_{i_j} \Delta v_{i_j} \sum_{\alpha,p} \lambda_p F_\alpha^{\,p}(\vec{r}_{i_j}, \vec{\xi})$$
$$= \sum_{s=0}^{3} \sum_i \beta_s n_{is} \sum_{\alpha,p} \lambda_p F_\alpha^{\,p}(\vec{r}_i, \vec{\xi}) \tag{59}$$

According to (45), the F_α^p depend on the index p only via the constant coefficient C_α^p, which means that the contraction $\lambda_p C^p$ (sum over p, other indices suppressed) should be proportional to the well-known term $\sum_p \lambda_p = \lambda$, compare Appendix F.[31] This leads to $\lambda_p C^p = C' \lambda$.

As next we make a further assumption which seems to be justified due to the symmetry of the 3-dimensional space: We presume that $\beta_s = \beta = \text{const}$, which allows to set $\sum_s \beta_s n_{is} = \beta N_i$ with new integer numbers N_i.

For the further evaluation the properties of the F_α have to be considered: As we have learned in Subsections 3.4.5 and 3.5.3, the condenser state functions have extrema at $K = \vec{\lambda}\vec{\xi} = \pm\frac{\pi}{2} m(2n+1)$ with parameter $m = 1$ or 2 and, according to Subsection 3.4.5, at $K = \vec{\lambda}\vec{\xi} = \pm 2\pi n$ for $r > 0$. The latter 'even' extrema were derived for the composition field, but are not valid for $F_{(3)}$. As already stated in Subsection 3.5.3, we assume that these 'eigenstates' are the states of the particle mass spectrum, which means for the expression in (59) that these extremal terms of F_α are to be inserted under the sum over index i. The dependency of the F_α on $\vec{\xi}$ is easily defined by the eigenvalues $K(n)$ via $\vec{\lambda}\vec{\xi} = K(n)$ and already led to the results (51) and (52) for the $F_{(3)}$. These still depend on $\vec{r}$. But now the relation $(\vec{\lambda}\vec{r})_{\text{ext}} = d \cdot (\vec{\lambda}\vec{\xi})_{\text{ext}}$ (the same holds with $\underline{\lambda}$) can be applied (see Subsection 3.5.2) if we interpret the states of the particle mass spectrum as the extrema of the F_α with respect to the variables $\vec{\xi}$ and $\vec{r}$. Therefore, we can reduce the sum over i in (59) to those contributions which fulfil the relation $C' \lambda \beta N_i = \tilde{C}(\vec{\lambda}\vec{\xi})_{\text{ext}} = \tilde{C} K(n)$ (with $\tilde{C} = \text{const}$).[32]

31 This assumption cannot be made without loss of generality. But as the constant coefficient C_α^p (currently) cannot be determined by the theory in general, our assumption, meaning the coefficient not being dependent on index p, seems to be the most trivial choice, which, as we shall see from our results, is supported by the phenomenology.

32 The condensation at the extrema should be proportional to an integer number N_i of condensed metrons and this can be measured by a projection of the metronic geometric structure onto a fixed "reference" (see [51] for an illustration of this issue) which obviously is the constant (coordinate independent) vector $\vec{\lambda} = \lambda \vec{a}^{-1}/q$, occurring in the solutions (32) and (45). Since $\vec{\xi} \neq 0$ generates the condensations (compare subsection 3.5.3), it is plausible that $\vec{\xi}$ is projected onto $\vec{\lambda}$ and provides the measure proportional to the N_i, and thus also proportional to λ. Hence $\lambda\beta N_i \sim (\vec{\lambda}\vec{\xi})_{ext} = K(n)$ follows.

Assembling these steps provides

$$E = \beta \sum_{i,\alpha} N_i \lambda F_\alpha(C'_\alpha, \vec{r}_i, \vec{\lambda}\vec{\xi} = K(n))$$
$$\to \sum_\alpha K(n) F_\alpha(\tilde{C}_\alpha, \vec{\lambda}\vec{r} = d \cdot K(n), \vec{\lambda}\vec{\xi} = K(n)) \tag{60}$$

where we have expressed that in the result the F_α have to be taken with an adjusted constant coefficient $\tilde{C}_\alpha$, depending on α (the coordinate indices $k, l = 4$ remain fixed). Note that in our above treatment the R_3 space volume effectively was transformed into a term proportional to ξ times a constant with dimension of an area, which seems only possible in our metron-based discretised R_6. It provides energy terms $\sim \vec{\lambda}\vec{\xi} = K(n)$ which are linearly proportional to a natural number, similar to a harmonic oscillator.

At this point we furthermore notice from (60) and the expressions $K(n)$ for the eigenvalues that only the positive branch in these expressions should be physically relevant, since an energy representing a particle rest mass should have a positive sign.

The further evaluation needs to consider the $F_\alpha(K(n))$ concretely. In principle, in the sum over the α in (60) all those condenser functions introduced in Subsection 3.5.3 and in Appendix H, references (158) and (159), have to be taken into account which contain the real partial metric structure $\kappa_{(3)}$ of the R_3 (as explained in Subsection 3.5.3). The $F_{(3)}$ and $F_{(123)}(d)$, in which $\vec{\xi}$ appears as full vector $\vec{\xi} = \vec{ct} + \vec{T}$, with $i\vec{T} = \vec{x}_5 + \vec{x}_6$, and no other coordinate dependency with $\vec{ct}$ or $\vec{T}$ is given (compare Appendix H, subsection H.5), belong to this group. These two functions can easily be calculated at the extrema $K(n)$, see subsequently. The function $F_{(13)}(c)$ corresponds structurally to $F_{(123)}(d)$, see reference (158), it only misses the dependency on $\vec{ct}$, as this variable is anti-hermetric in the form c. But our formalisn obtained so far can be applied to this form, we just have to realise that $\vec{\xi} = \vec{T}$ in $K(n) = \vec{\lambda}\vec{\xi}$ in this case. This means that there is no difference between $F_{(13)}(c)$ and $F_{(123)}(d)$ in our formalism, i.e. the difference between uncharged and charged particles which we associate with the hermetry forms c and d cannot be expressed by it. Different energy levels due to this difference cannot be resolved. These may come into play through "second order" effects as influence, respectively, interaction by the form b (electromagnetic field) and should be subject to further analysis. The remaining relevant F_α are $F_{(13)}(d)$ and $F_{(23)}(d)$, in which one of the variables $\vec{P} = i\vec{ct}$ or $\vec{P} = i\vec{T}$ appears in the term $e^{-\vec{\lambda}\vec{P}}$. For this respective term the eigenvalues $K(n)$ of the extrema cannot be inserted. But a calculation in Appendix H, subsection H.5, shows that these $F_\alpha(\vec{P})$, i.e. their relevant real part, depend on the $\vec{P}$ in a way that the condenser function fluctuates around zero, i.e. gives a contribution of 0 in average.

Therefore, we restrict the further evaluation of the sum in (60) to the functions $F_{(3)}$ and $F_{(123)}(d)$. From (60) and the expressions of these functions according to (51), (52) and (159) we see that the terms $(\vec{\underline{\lambda}}\vec{r})_{\text{ext}}$ and $(\vec{\lambda}\vec{r})_{\text{ext}}$ have to be expressed through $(\vec{\lambda}\vec{\xi})_{\text{ext}} = K(n)$ etc. to achieve results reduced to the eigenvalues. This is possible with the introduced relations $(\vec{\lambda}\vec{r})_{\text{ext}} = d \cdot (\vec{\lambda}\vec{\xi})_{\text{ext}}$ and with the eigenvalues (defined at the extrema which we consider) $(\vec{\lambda}\vec{\xi})_{\text{ext}} = \beta_{\pm}$, both also given for $\underline{\vec{\lambda}}$ and $\underline{\beta}_{\pm}$:

$$
\begin{aligned}
&(\vec{\lambda}\vec{r})_{\text{ext}} = \beta_{\pm} d, \quad \beta_{\pm} = K(n) = \frac{\pi}{2} m(2n+1),\\
&\qquad m = 1,2 \quad \text{or} \; = \frac{\pi}{2} m(2n), \; m = 2\\
&(\underline{\vec{\lambda}}\vec{r})_{ext} = \underline{\beta}_{\pm} d, \quad \underline{\beta}_{+} = \frac{\pi}{2}(2\underline{n}_{+} + 1),\\
&\qquad \underline{\beta}_{-} = \pi \underline{n}_{-} = \frac{\pi}{2}(2\underline{n}_{-}), \quad \Rightarrow \quad \underline{\beta}_{\pm} = \frac{\pi}{2}\underline{n}
\end{aligned}
\tag{61}
$$

Here we have applied the expression for the eigenvalues $K(n)$ (positive branch). The eigenvalues β_+ are given by $m = 1$, the β_- by $m = 2$. The $\underline{\beta}_{\pm}$ can be summarised in one expression, as they do not have to be differentiated in the further evaluation of the F_α. With (61) and (47) at hand, the $F_{(3)}$ of (51) and (52) at the extrema and with the redefined constants $\tilde{C}_\alpha$ can be calculated as

$$
\begin{aligned}
m = 1: \quad F_{(3)\,\text{ext1}} &= \tilde{C}_{(3)}\, e^{\frac{\pi}{2}\underline{n}d}\\
&\quad \times \left(\frac{1}{2}(1 + e^{\pi(2n+1)d})\right)^{-\frac{n}{2(2n+1)} f_{\text{ext}}^{-1}}\\
m = 2: \quad F_{(3)\,\text{ext2}} &= \tilde{C}_{(3)}\, e^{\frac{\pi}{2}\underline{n}d}\\
&\quad \times \left(\frac{1}{4}(1 + e^{\pi(2n+1)d})^2\right)^{-\frac{n}{4(2n+1)} f_{\text{ext}}^{-1}}
\end{aligned}
\tag{62}
$$

The f_{ext} are the values of function $f = f(k, l = 4)$ at the $\vec{x}$-extrema of the F_α, compare Subsection 3.5.2. In that numerical range of d and f which becomes relevant to meet empirical mass data the expressions above can be well approximated (error of ≤0.002% for $n > 0$) by

$$
F_{(3)\,\text{ext}} = \tilde{C}_{(3)}\, 2^{\left(\frac{n}{2(2n+1)} f_{\text{ext}}^{-1}\right)}\, e^{\frac{\pi}{2}\underline{n}d\,(1 - f_{\text{ext}}^{-1})}. \tag{63}
$$

For the calculation of $F_{(123)\text{ext}}$ we can use the results of (45) and (61) and continue with

$$
\begin{aligned}
&F_{(123)\,\text{ext}}/\tilde{C}_{(123)}\\
&\quad = e^{\underline{\vec{\lambda}}(\vec{r}+i\vec{\xi})_{\text{ext}}} \left(e^{\vec{\lambda}(\vec{r}+i\vec{\xi})_{\text{ext}}} - 1\right)^{-\alpha}\\
&\quad = e^{\frac{\pi}{2}\underline{n}(d+i)} \left(e^{K(n)(d+i)} - 1\right)^{-\alpha}\\
&\quad = e^{\frac{\pi}{2}\underline{n}d} \left(\cos\left(\frac{\pi}{2}\underline{n}\right) + i\sin\left(\frac{\pi}{2}\underline{n}\right)\right)\\
&\qquad \times \left[e^{K(n)d}(\cos K(n) + i \sin K(n)) - 1\right]^{-\alpha}.
\end{aligned}
\tag{64}
$$

After a simplification of the term in the square bracket as given in Appendix I and with use of the trigonometric addition theorems we get the approximate result (error <0.2% for $n = 1$ and <0.004% for $n \geq 2$)

$$
\begin{aligned}
&F_{(123)\,\text{ext}}/\tilde{C}_{(123)}\\
&\quad = e^{\frac{\pi}{2}\underline{n}d\,(1-f_{ext}^{-1})} \left(\cos\left(\frac{\pi}{2}\underline{n}\left(1 + (2-m)\frac{(-1)^n}{2n+1} f_{\text{ext}}^{-1}\right)\right)\right.\\
&\qquad \left. + i \sin\left(\frac{\pi}{2}\underline{n}\left(1 + (2-m)\frac{(-1)^n}{2n+1} f_{\text{ext}}^{-1}\right)\right)\right)\\
&\quad \to \; e^{\frac{\pi}{2}\underline{n}d\,(1-f_{\text{ext}}^{-1})} \left(\cos\left(\frac{\pi}{2}\underline{n}\right) + i\sin\left(\frac{\pi}{2}\underline{n}\right)\right) \quad \text{for}\\
&\qquad n \gg 1 \; (m = 1) \; \text{or} \; m = 2\\
&\quad = e^{\frac{\pi}{2}\underline{n}d\,(1-f_{\text{ext}}^{-1})} (-1)^{\frac{n}{2}} \qquad \text{if } \underline{n} \text{ is even,}\\
&\quad = e^{\frac{\pi}{2}\underline{n}d\,(1-f_{ext}^{-1})}\, i(-1)^{\frac{n-1}{2}} \quad \text{if } \underline{n} \text{ is odd.}
\end{aligned}
\tag{65}
$$

Note that the result puts a selection rule on the number $\underline{n}$ at $m = 1$ if $n \gg 1$ and for $m = 2$, as then the real part of $F_{(123)\text{ext}}$, which is relevant for the mass contribution, is $\neq 0$ only for even $\underline{n}$. For these the imiginary part vanishes. This is to be discussed when it comes to lifetimes.[33]

With these results the calculation of the energy (mass) of (60) can be finalised. The sum over the partial structure terms α reduces to the terms of $F_{(3)\,\text{ext}}$ and $F_{(123)\text{ext}}$ and only the real part of these functions has to be taken:

$$
\begin{aligned}
E = \frac{\pi}{2} mN &\left[\tilde{C}_{(3)} \frac{1 - (-1)^N}{2}\, 2^{\left(\frac{n}{2N} f_{\text{ext}}^{-1}\right)}\right.\\
&\left. + \tilde{C}_{(123)} \cos\left(\frac{\pi}{2}\underline{n}\left(1 + (2-m)\frac{(-1)^n}{N} f_{\text{ext}}^{-1}\right)\right)\right]\\
&\times e^{\frac{\pi}{2}\underline{n}d\,(1-f_{\text{ext}}^{-1})} \qquad \text{for } N = 2n+1 \text{ and } m = 1,2
\end{aligned}
$$

33 We see from the results of this subsection that a non-zero imaginary part and thus a finite lifetime only exists for $F_{(123)\text{ext}}$ and $m = 1$ (for even $\underline{n}$ which provide non-vanishing real parts, i.e. masses), which does not meet the empirical results. Concrete values which could be calculated from these non-zero contributions (with the $F_{(123)}$ contribution in Eq. (66) and a sine instead of a cosine expression in it and the relation for the lifetime $\tau = \hbar/\Gamma$, with $\Gamma = \text{Im}(E)$) are far to small (order of magnitude $10^{-23}s$). This is obviously due to the fact that our theory cannot yet model the elementary interactions (including their coupling constants), responsible for the decays. The only qualitatively true result can be seen for the leptons, the electron is stable (zero imaginary part due to $\underline{n} = 0$), the muon and the tau decay. Here the imaginary energy contribution Γ amouts to about 50−55 MeV.

$$\text{or} \quad N = 2n \text{ and } m = 2 \tag{66}$$

In (66), we have added the resulting expressions of two partial structures, which in general differ in their structure dependent quantities. As these resulting expressions contain the same numbers $m, n, \underline{n}$, their potential remaining difference reduces to the functions f_{ext} and d. So in general these would have to be indexed with '(3)' and '(123)' per term in (66). But we assume that they do not differ between the partial structures and shall see in the next subsections that a very good agreement with the empirical data can already be achieved this way, with f_{ext} assumed to be constant, i.e. independent of the quantum numbers. This approach also implies that the above mentioned selection rule that only even $\underline{n}$ yield non-vanishing values has to be applied to the $F_{(3)}$ contribution as well, i.e. to the whole formula (66).

But first we consider the behaviour for $n \gg \underline{n}$. Formula (66) then becomes:

$$E \to \frac{\pi}{2} mN \left(\tilde{C}_{(3)} \frac{1-(-1)^N}{2} + \tilde{C}_{(123)} (-1)^{\frac{n}{2}} \right) \mathrm{e}^{\frac{\pi}{2} \underline{n} d \left(1 - f_{\text{ext}}^{-1}\right)} \tag{67}$$

We find that in the relevant numerical range equation (67) differs less than 2% from (66) for $n \geq 8$ ($F_{(3)}$ contribution), respectively, for $n \geq 4$ ($F_{(123)}$ contribution) (for $\underline{n} = 2$). This means that (67) is a good approximation for the area of mid and higher n, but not exact enough for the lowest n. Another important aspect is the question, how the coefficients $\tilde{C}_\alpha$, here $\alpha =$ '(3)' or '(123)', have to be treated, i.e. whether they are global constants or depend on certain indices. We have made clear in Appendix H (subsection H.5) that the exponent $\underline{\alpha}$ and the function f in general differ between the various condenser functions F_α. As each $\underline{\alpha}$ depends on the index $\underline{n}$ (see (47) and (61)), which parametrises (in integer steps) the condensation strength, it can be assumed that in a superposition of $F_\alpha(\underline{n})$ the coefficients differ not only between the different F_α, but between the $\underline{n}$ per F_α as well and also depend on the parameter m and on the "parity" $P = (-1)^N$, see above, thus $\tilde{C}_\alpha = \tilde{C}_\alpha(m, P, \underline{n})$. We shall use this assumption and notation subsequently, but no further dependency of the $\tilde{C}_\alpha$.

4.2 Determination of unknown functions – link to empirical data

Next we have to consider the properties of the functions d and f, both at the extremal points of the F_α (d is only defined at these). For this we realise that the exponent $\frac{\pi}{2} \underline{n} d \left(1 - f_{\text{ext}}^{-1}\right)$ should be positive, as it obviously describes an increase in the energy spectrum in which $\underline{n} = 0$ marks the lowest level of a set of states differentiated by the n (in $\frac{\pi}{2} mN$). From this positive exponent we conclude that the function d should be positive and the term $1 - f_{\text{ext}}^{-1}$ as well.

Formulae (66) and (67) provide the following results for the lowest $\underline{n}$ and certain n for the odd ($N = 2n+1, P = -1$) and for the even spectrum ($N = 2n, P = 1$) which can be related to the masses of the electron, the tau and pion and be used to adjust the so far unknown functions and parameters to empirical data (we use the convention $E = M$, i.e. $c = 1$):

$$\begin{aligned}
E_{m=1,P=-1,\underline{n}=0,n=0} &= \frac{\pi}{2} \left(\tilde{C}_{(3)}(1,-1,0) \right. \\
&\quad \left. + \tilde{C}_{(123)}(1,-1,0) \right) := m_{\mathrm{e}}, \\
E_{m=1,P=-1,\underline{n}=2,n=25} &\approx \frac{51\pi}{2} \left(\tilde{C}_{(3)}(1,-1,2) \right. \\
&\quad \left. - \tilde{C}_{(123)}(1,-1,2) \right) \mathrm{e}^{\pi d \left(1 - f_{\text{ext}}^{-1}\right)} \\
&:= m_\tau, \\
E_{m=2,P=1,\underline{n}=2,n=1} &= -2\pi \tilde{C}_{(123)}(2,1,2) \, \mathrm{e}^{\pi d \left(1 - f_{\text{ext}}^{-1}\right)} \\
&:= m_\pi.
\end{aligned} \tag{68}$$

The second expression (according to (67)) holds approximately (error <0.5%). Apparently, these results lead to an appropriate scheme if we set

$$\begin{aligned}
&\tilde{C}_{(3)}(1,-1,0) + \tilde{C}_{(123)}(1,-1,0) = \frac{2m_{\mathrm{e}}}{\pi}, \\
&\tilde{C}_{(3)}(1,-1,2) - \tilde{C}_{(123)}(1,-1,2) = \frac{2m_{\mathrm{e}}}{\pi} \\
&\qquad \to \quad \mathrm{e}^{\pi d \left(1 - f_{\text{ext}}^{-1}\right)} \approx \frac{m_\tau}{51 m_{\mathrm{e}}} \approx 68.2, \\
&-\tilde{C}_{(123)}(2,1,2) = \frac{2m_e}{\pi} \\
&\qquad \to \quad \mathrm{e}^{\pi d \left(1 - f_{\text{ext}}^{-1}\right)} = \frac{m_\pi}{4 m_{\mathrm{e}}} \approx 68.5.
\end{aligned} \tag{69}$$

In the last row there is no $\tilde{C}_{(3)}$ contribution since it does not exist for the even eigenvalues $K(n)$, as we saw before. The numercial value for the $\mathrm{e}^{\pi d \left(1 - f_{\text{ext}}^{-1}\right)}$ term in this row was obtained by inserting $m_\pi \approx 140$ MeV/c^2, which corresponds to the phenomenological rule of Mac Gregor, compare Section 1. Both calculated values are close to half of the inverse fine structure constant, $1/2\alpha = 68.518$, which we shall analyse further in the next section. This allows to derive a value for f_{ext} in terms of order of magnitude if we assume that d varies only weakly with ξ and lies in the range of 0.5–1 (see below): $f_{\text{ext}} \approx -0.6$ for $d = 0.5$ and $f_{\text{ext}} \approx -2.9$ for $d = 1$ (a limiting value which must not be reached, see below). It has to be checked, whether this result can be consistent with the finding of (53) et sqq., stating that $f = f(k, l = 4)$ should become positive and > 1 for large r. The definition of f in Appendix H, (144), suggests that its coordinate-dependent part $\tilde{f}$ can vary strongly, depending on the relation of the coordinates x_l, x_m to each other. If we assume x_l and x_m, i.e. all coordinates, thus r and ξ, being in the same order of magnitude for small r, which should hold for the area of condensations creating mass and which corresponds to our assumption $0.5 \leq d < 1$, then the approximate formula for $\tilde{f}$ of Appendix H (below (143)) and hence for f becomes applicable and allows to calculate pairs of coefficients $\underline{a}(k, l = 4), a(k, l = 4)$. With these coefficients, $\tilde{f}$ and f can be calculated for the limit of large r, i.e. $x_m \gg x_l$ in (143) and this indeed yields $f > 1$ and so a negative value for $1 - f$, as necessary for a asymptotically vanishing function F^i_{44}.[34] Thus, the ansatz to assume $f = f_{\text{ext}}$ being constant in the range of the said negative value for the considered area of condensation seems to be compatible with the expected asymptotic behaviour.

In order to find an approach for the function d we first remember that $r < \xi$, i.e. $d < 1$ must hold in the area of discrete eigenvalue spectra (compare (33) et sqq.). The question is how d concretely appears at these condensation extrema $K = \vec{\lambda}\vec{\xi} = \frac{\pi}{2} m(2n + 1)$ (respectively $K = 2\pi n$) which determine the relevant points in the R_6 for the mass spectrum in our model. We recall the relation (36) between r and ξ which was obtained for these extrema, and realise on the other hand that the starting point for it was the integer dependency $2n + 1$ (or $2n$) of the ξ-related eigenvalues, and that an analogous integer form is possible for r (under the conditions explained below (35)), but *not necessary* and could be a too strong restriction. However, the relation (36) provides a means to model d if we can eliminate the integer variable n_r as independent parameter, as then an expected nearly linear course of the relation $r - \xi$, i.e. a nearly constant d can be achieved. Our ansatz is to set $n_r = n/s$ with real parameter $s > 1$ so that $n_r < n$ and $r = \frac{2n/s+1}{2n+1}\xi < \xi$, thus $d = \frac{2n/s+1}{2n+1} < 1$ ($d = 1$ would only occur for $n = 0$ which is not relevant in the spectra for $\underline{n} > 0$ where d becomes active, see (66) and Table 1). For the even K we get $d = \frac{n/s}{n} = \frac{1}{s} < 1$. The parameter s ultimately has to be considered as a fit parameter, but should lie in a range which is compatible to the given constraints and with kinematic considerations (see Appendix H, footnote 57).

[34] Concrete exemplary data which provide a realistic value of $f = -2.15$ for $x_m \approx x_l$ (core area of condensation) and which yield $f = 1.74(1.59)$ in the asymptotic area $x_m = 10x_l(100x_l) \gg x_l$ are $\underline{a}(k, l = 4) = 2.0, a(k, l = 4) = 1.366, q = 3$.

It should be mentioned that an even simpler approach for d as $d = \text{const} < 1$ cannot be excluded from the start. But calculations with such a constant d do not provide results better than 0.89% error to the empirical mass data on average (for the subset of particles listed in Table 1 of the next subsection), so worse than the results presented in the following. In particular, the mass of the muon is underestimated with 98 MeV in this scenario. This supports our above approach derived from (36).

4.3 Results

Assembling the results of the last two sections, we can derive a concrete formula for calculating the mass energies if we succeed in reducing the ambiguity given by the dependency on the parameters $\tilde{C}_\alpha(m, P, \underline{n})$ which the theory cannot determine (which results from the fact that its fundamental equation (28) is homogeneous, different from the Einstein equation). The relations of (69) give a good hint, how this dependency can be shaped with a minimal remaining degree of freedom: For the even eigenvalues ($K(n) \sim 2n, m = 2, P = 1$) the $\tilde{C}_{(123)}(2, 1, \underline{n})$ can directly be determined by m_e through $\tilde{C}_{(123)}(2, 1, \underline{n}) = \frac{2m_\text{e}}{\pi}(-1)^{\frac{n}{2}}$, as there is no dependency on the $\tilde{C}_{(3)}$ in this case. The sign-factor ensures positive energy values. For the odd eigenvalues ($K(n) \sim 2n + 1, P = -1$) there is a superposition of $\tilde{C}_{(3)}$ and $\tilde{C}_{(123)}$ terms, which for our two "fit" masses m_e and m_τ add equally balanced (disregarding the sign). Since the $\tilde{C}_\alpha$ remain undetermined by the theory, we must adjust them to the empirical values in a way so that the mass scale is fixed on the one hand, but that the existing freedom in the weight between the $F_{(3)}$ and the $F_{(123)}$ contributions is expressed by a remaining parameter on the other hand. We do this by introducing the parameter C with $0 \leq C \leq 1$ which determines the strength of the $\tilde{C}_{(3)}$ in the superposition of the $F_{(3)}$ and the $F_{(123)}$ terms in (66) and (67) globally. Furthermore we take into account that the energies have to be positive and that obviously only a superposition of positive partial contributions can add to a phenomenologically right spectrum. This leads to the approach $\tilde{C}_{(3)}(m, -1, \underline{n}) = \frac{2m_\text{e}}{\pi} C$, $\tilde{C}_{(123)}(m, -1, \underline{n}) = \frac{2m_\text{e}}{\pi}(-1)^{\frac{n}{2}}(1 - C)$ for the odd spectrum. Thus, we obtain the following formula for the energy E in continuation of (66)

$$E = m_\text{e} m N \left[C\, 2^{\left(\frac{\underline{n}}{2N} f_{ext}^{-1}\right)} + (1 - C) \left| \cos \left(\frac{\pi}{2}\underline{n} \left(1 + (2 - m)\frac{(-1)^n}{N} f_{\text{ext}}^{-1} \right) \right) \right| \right] \mathrm{e}^{\frac{\pi}{2}\underline{n} d \left(1 - f_{\text{ext}}^{-1}\right)} \qquad (70)$$

for odd terms $N = 2n + 1, m = 1, 2$ and $d = \frac{2n/s+1}{2n+1}, 0 \leq C \leq 1$, or for even terms $N = 2n, m = 2, d = \frac{1}{s}, C = 0$. The absolute value of the cos-term expresses the effect of the sign-factors of the chosen $\tilde{C}_{(123)}$ coefficients, ensuring positive values. It differs from 1 only for $m = 1$. We conclude from this that, as a calculation rule, the absolute values of the condenser functions $|F_\alpha|$ have to be taken. For $n \gg \underline{n}$ formula (70) generally takes the form which holds for the even spectrum ($N = 2n, m = 2$) exactly

$$E \to m_e mN\, e^{\frac{\pi}{2}\underline{n}\,(1-f_{\text{ext}}^{-1})/s} \tag{71}$$

and merges from a formula with four adjustable parameters in (70), being m_e, f_{ext}, s and C, to an expression with effectively the two parameters m_e and $a = \left(1 - f_{\text{ext}}^{-1}\right)/s$. The running "quantum numbers" m, N and $\underline{n}$ instead, just as the structure of the formulae, emerge directly from the theory. When we use the free parameters to adjust to the empirical mass data, we find a best fit with the following numerical values: For m_e we of course have to insert the electron mass of 0.511 MeV/c^2. f_{ext} and s become $f_{\text{ext}} = -2.1573$ and $s = 1.089$, i.e. $\pi a = \pi\left(1 - f_{\text{ext}}^{-1}\right)/s = 4.2221$ and $\exp(\pi a) = 68.175$. The last value is relatively close to the half of $1/\alpha$, the inverse of the fine structure constant, $1/2\alpha = 68.518$, which is not surprising, as we have found the same structure of relations for the most considered masses as Mac Gregor did and as Hansson's approach (6) and our subsequent derivations of Section 2 suggested:

For $m = 1, \underline{n} = 0$ and $n = 0$, i.e. $N = 1$ we get $E = m_e$ in (70) and (71), which simply reflects the fixing of the mass scale by setting the coefficients $\tilde{C}_\alpha$ as defined above. $m = 1, \underline{n} = 2$ in (71) gives $E \approx m_e(2n+1)\exp(\pi a) \approx m_e(2n+1)/2\alpha$, which provides $m_\mu \approx \frac{3}{2}m_e/\alpha$ ($n = 1$), $m_\tau \approx \frac{51}{2}m_e/\alpha$ ($n = 25$) and $m_{\text{u,d}} \approx \frac{9}{2}m_e/\alpha$ ($n = 4$) for the constituent u, d quarks and $m_N \approx \frac{27}{2}m_e/\alpha$ ($n = 13$) for the nucleon (p, n) in the hadronic sector.

$m = 2$ and the odd spectrum $N = 2n+1$ provide $E \approx 2m_e(2n+1)\exp(\pi a\underline{n}/2) \approx 2m_e(2n+1)(1/2\alpha)^{\underline{n}/2} = m_e (2n+1)/\alpha$ for $\underline{n} = 2$ and $= \frac{m_e}{2}(2n+1)/\alpha^2$ for $\underline{n} = 4$. The latter formula yields a $m_{\text{gb}} \approx \frac{9}{2}m_e/\alpha^2$ for $n = 4$, which relates it to the $m_{\text{u,d}}$ by $m_{\text{gb}} = m_{\text{u,d}}/\alpha$. So, m_{gb} is identical to Mac Gregor's "gauge boson" mass, compare Section 1. Finally, for $m = 2$ and the even spectrum $N = 2n$ we find $E = 2m_e\, 2n\, \exp(\pi a\underline{n}/2) \approx 4m_e n(1/2\alpha)^{\underline{n}/2}$, which is $= 2m_e n/\alpha$ for $\underline{n} = 2$ and provides $m_\pi \approx 2m_e/\alpha$ ($n = 1$), or $= m_e n/\alpha^2$ for $\underline{n} = 4$.

The accordance of these resulting formulae with the phenomenologically derived expressions of references [10, 11, 17, 18] and with our derivation (11) in Section 2 is one of the most interesting achievements of this work.

The numerical results obtained with the expressions (70) and (71) are presented in Table 1 for a selected subset of particles, including the three leptons[35] and those light hadrons which may be most interesting in a first step to evaluate the accordance with the empirical mass data. The complete calculation for (nearly) all known particles and resonances is given in Appendix N (Table 2). The empirical mass data presented in the tables were taken from [1]. Before we evaluate these results, we should become aware of the constraints for deviations between the calculated mass spectrum and the empirical data.

With the calculated formulae, containing the even and odd eigenvalue terms, we have obtained a mass spectrum in the shape of "grids" with a nearly constant distance between the consecutive values per grid. The number mN in (70) and (71) "counts" the positions in the grid, the quantity $\underline{n}$ in the exponent of the e-function determines the size of the distance for different levels, i.e. grids: m_e for the lowest level $\underline{n} = 0$, ≈ 35 MeV for the second level $\underline{n} = 2$ and ≈ 2400 MeV for $\underline{n} = 4$ (odd values are "forbidden", see (65) et sqq.). The lowest value and the distances in the grids are determined by our fit to the empirical data as decribed above (by setting the parameters m_e, f_{ext} and s as defined above) and by the observation that the 35 MeV distance (of the second level) describes a set of the lower particle spectrum very well, similar as observed in [10–27].

This being put in place, we can compare the empirical particle masses with our grid values, but cannot assign a single particle to a certain value in the grid, as our theory does not (yet) provide an assignment of the known quantum numbers like spin, isospin, strangeness etc. to the mass-energy grid values. Instead we can allocate the best fitting grid value, determined by the integer numbers $m, N(n), \underline{n}$ of our theory, to each empirical particle mass. In this way Tables 1 and 2 have been created. A consequence is that the error, the deviation between the calculated and the empirical mass value, has a maximum and an average value even for a radom empirical mass value: The maximum is given by the half grid size, e.g. $\approx$ 35/2=17.5 MeV for $\underline{n} = 2$, the average error by a quarter, i.e. ≈ 8.75 MeV for $\underline{n} = 2$. We call this average error for a random set of masses the "statistical" error and compare the errors, calculated for our theoretical results, against it, since it provides the measure to judge about the quality and predictive power of the theory when calculating a mass spectrum. This "statistical" error Δ_s is 1.29% for the data set of Table 1 and 0.49% for the full set of Table 2 in Appendix N. The lower value of the full set is of course due to the much larger amount of bigger

35 Without the neutrinos, see Section 5.

masses in this set, while the absolute average error remains constant.

We note here that the statements made above have to be qualified concerning the assignability of a spin: Our theory can derive a spin from the metron picture, these being oriented elementary areas with angular momentum-like properties, see subsection 3.5.1 and Appendix K. However, the connection of the ingredients of our mass formula to a spin is not clear. Nambu and later other authors who analysed the empirical data (see e.g. [18, 23]) concluded from the observed patterns that the mesons (as bosons) obviously are assigned to an even quantum number in Nambu's formula, i.e. the number mN in our formulae, and leptons and baryons to an odd number. Therefrom they derived the picture of constituents (as in the constituent quark model) or partons of a mass unit of 35 MeV which the particles consist of and which carry spin 1/2.

If we follow this picture, then the parameter $m = 1, 2$ in our model defines, whether an even or odd number of partons makes up the particle and thus whether its spin is integral or half-integral. For the systematics described above, how to assign a mass value of the "grids" to a single particle, this means that values with even quantum number ($m = 2$) can only be assigned to bosons and values with odd quantum number ($m = 1$) only to fermions in this picture. We have conducted such a different classification in Appendix N, Table 3. As the distance between two "allowed" neighbouring mass values then doubles to 70 MeV, the "statistical" error doubles as well (to 17.5 MeV for $\underline{n} = 2$) and the real errors increase roughly be the same factor (see Table 3). Since this "parton number – spin" relation (as we want to call it) is only a conjecture and not without alternatives,[36] we regard it only as one possible model variant and first consider the less restrictive variant introduced above.

When we therefore first consider the errors of the calculations for the full set of Table 2, we see that the average percentaged error over all particles lies between 0.43% (calculations E_1 and E_2) and 0.53% (calculation E_4) and thus is only slightly better than the "statistical" error of 0.49%, or even a bit worse (E_4). This shows that our approach unavoidably looses significance against randomness for high mass values (compared to the grid size), since the calculated mass grid then becomes too fine-meshed.[37] Nonetheless the calculations reach a very low error to the empirical masses, which means that the error is less than 0.5% for more than 63% of the particles (in calculations $E_1 - E_3$, 53% for E_4), less than 1% for 90% of the particles and less than 2% for more than 99% of them. The only particles with a higher error than 2% are the π^0 with 3.24% (resp. 3.76% in E_4) and the η, but the latter only in calculation E_4 with 2.25%.

Comparing these results against the calculation of Sidharth [29] who, although coming from a different approach, also obtained a "grid"-like result without assignability of the physical quantum numbers as spin, isospin etc., our results show noticeably less deviation from the empirical masses.[38] According to his resulting formula $E = m_P = m(n + 1/2)m_\pi$ with natural numbers m, n, his model provides only masses in a subset of our results, as $m_P = 2m(2n + 1)m_\pi/4 \approx 2m(2n + 1)$ (35 MeV) in our result structure. It misses the odd terms for our parameters $m = 1, N = 2n + 1$, i.e. $E \approx (2n + 1)$ (35 MeV), thus it cannot well reproduce the masses of particles which obviously belong to this number pattern like the nucleon (p, n) and a lot of other baryon and meson states, see Table 2 (the μ and τ also fall into this odd pattern in our theory, but as leptons they cannot be covered by Sidharth's model, since it bases on a QCD derived approach). It nevertheless seems remarkable that our model, in a subset, obviously produces a result set which contains the results of a QCD based model. Varlamov's approach [7] instead, as already mentioned in Section 1, covers the leptonic and hadronic sector of particles with an excellent average accuracy of 0.41% and can assign a spin per particle state. Though, our result E_1 reproduces the masses of some particles of particular interest, as the $\mu, \tau, N, \pi^\pm$ and ρ, slightly better (0.36% in average for these particles) than his results do (0.72%). Interesting in his approach is, similar to ours, that the formulae of Nambu and Sidharth (latter approximately) can be brought into Varlamov's form. Then the formally calculated spin does not make sense, however.

36 Since in Heim's theory the elementary spin results from the single metrons, it is conceivable that there could be constituents/partons, i.e. amounts of metrons, with zero or, more general, integral spin, instead of 1/2 spin. Secondly, note that Nambu made his assumption of bosons having even and fermions having odd numbers of 35 MeV mass units on the basis of only 8 particles known in 1952. The overall set of particles known today does not show such a clear picture in this regard.

37 Also Greulich pointed to the fact that an accurate calculation of heavy masses via an Eq. (1) with large N is trivial [27].

38 Sidharth achieved an error of less than 1% for 63% of the particles, less than 2% for 93% percent of the particles, and less than 3% for all particles with the sole exception of the $\omega(782)$, in which case the error was 3.6%. His list of particles was (of course) not fully identical to ours, but so far in agreement that the results can be compared on this statistical level.

We now consider the results in Table 1 with the selected subset of particles. Most importantly, it shows that the average error of the calculated masses compared to the empirical data is clearly lower than the "statistical" error. The ratio Δ/Δ_s to it ranges from 0.52 (calculation E_1) to 0.59 (calculation E_4), so the calculated masses for this subset clearly have predictive power (due to the fact that these masses are not too high compared to the grid size, as already explained above).

A comparison of the four different calculations $E_1 - E_4$ shows that E_1 and E_2 give nearly identical good results, both calculated according to Eq. (70), i.e. our result with high accuracy also for low N, and differing only in the parameter C which determines the weight of the $F_{(3)}$ contribution. Calculation E_1 with $C = 0.9$ provides a very slightly lower error for the particle subset of Table 1 than E_2, which on the other hand is for a tiny amount better in the error for the whole particle set in Table 2 (can only be seen in the minimally lower ratio to Δ_s, in the error itself only in the third decimal place). The admixture of the $F_{(123)}$ contribution in E_1 is low, only 10%. It was obtained by varying C and finding that $C = 0.9$ provides the lowest error value for the set of Table 1. That C lies so close to 1 shows that the $F_{(3)}$ contribution dominates the mass energy for the odd eigenvalues $N = 2n + 1$ ($m = 1, 2$). In the even case ($N = 2n, m = 2$) there is no $F_{(3)}$ contribution at all (see (70), i.e. $C = 0$), only $F_{(123)}$, so that the calculated masses do not differ between E_1 and E_2 for this case. The effect of the $F_{(123)}$ contribution for the odd terms can be illustrated by indicating the masses for the nucleon and the ρ meson if calculated purely with $F_{(123)}$: 952.40 and 793.05 MeV. Thus, the $F_{(123)}$ contribution provides noticeably higher values than the $F_{(3)}$ (which is purely given in calculation E_2 for the odd terms).

Calculation E_3 according to Eq. (71) was carried out to test the quality of this approximation. As to be expected, it delivers noticeably worse results for the subset of particles in Table 1 (average error of 0.75%), which lie in the lower range of the overall spectrum and therefore contain also terms with low N, for which the approximation (71) is not very good. Overall it underestimates the masses in this range compared to the "exact" calculations E_1, E_2, which in few cases leads to even closer agreement to experiment (nucleon, τ), but provides too low values for e.g. the Δ, Σ, K, ρ and $\omega(782)$ particles. Note that it gives identical results as E_1, E_2 for the even terms ($N = 2n$) because the $|\cos()|$ term in (70) becomes equal 1 in this case without approximation. It is remarkable that calculation E_3, despite the comparatively worse reproduction of the data, still provides a good description with a formula containing effectively only two parameters, m_e and $a = \left(1 - f_{\text{ext}}^{-1}\right)/s$ (the last term can be considered as one parameter, since s and f_{ext} appear only once at this point in (71)). Calculations E_1, E_2 contain a more complex dependence on them and additionally on parameter C, so they depend on four parameters in this sense.

Finally, we consider calculation E_4, which was performed according to Eq. (71) et sqq. with $\exp(\pi a) := \frac{1}{2\alpha} = 68.518$, i.e. as an approximation which corresponds to the phenomenological approach of Section 2 and so, in a subset, is fully identical to the Nambu formula (1). The difference to E_3 consists only in the slightly different value in the exponent of formula (71). The results show a still little bit higher error (0.76%) than E_3 for the set of Table 1, but a clearly larger error than E_3 (and so E_1, E_2) for the whole particle set of Table 2. Also for single values like the masses of the τ, the nucleon and the $\Omega(1672)^-$ calculation E_4 provides the worst match with experiment compared to the other three calculations. But, as already stated above, its remarkable value is that it demonstrates the obvious correspondence between the phenomenological approach of Section 2 and our theoretical ansatz based on Heim's theory. It shows that the inverse of the fine structure constant, as a factor between the strengths of the electromagnetic and the strong interaction, obviously plays the role of boosting the mass scale, as suggested by the cited references (first boost in terms $\sim \frac{1}{\alpha}$ for $\underline{n} = 2$, second boost in terms $\sim \frac{1}{\alpha^2}$ for $\underline{n} = 4$).

We now shortly evaluate the results given in Appendix N, Table 3, in which we applied the "parton number – spin" relation, we explained above. We clarified already that in this case the "statistical" and real errors are around twice the values of calculations of Table 2. It should be noted that the results for several particles appear to be much worse than in the scenario of Tables 1 and 2, especially for the Λ, Σ, Ξ and the Ω^- baryons (incl. many of their resonances). The very good agreement with the experimental data for these particles in this first considered scenario suggests that there should be a physical mechanism to allow for even numbers of mass units (partons) also for fermions.

At this point it is time to draw a conclusion about the hierarchical structure of the "grid" levels $\underline{n}$. It obviously divides the particles into three groups according to their mass, the electron alone makes up the first group (level $\underline{n} = 0$), all particles in the energy range from ~100 MeV to ~10,000 MeV fall in the group of the second level ($\underline{n} = 2$) and the currently known heavy particles with masses of > 10,000 MeV like the massive gauge bosons, the top quark and the Higgs obviously belong to a third group (level

$\underline{n} = 4$). An exact energy threshold for the levels cannot be given, but if the overall model shall have predictive power, it seems to be clear that the high masses in the range of gauge bosons and the top quark must belong to the third level. Empirically, there is a big gap between the heaviest known (bottom and charm flavoured) mesons and baryons and these very heavy particles, which supports this view.

Up to now we have purposely avoided to name the three levels "generations", because the grouping which is clearly suggested by our theory does not comply with the generations defined or assumed in particle physics according to the SM so far. There, each of the leptons together with its neutrino is combined with two quarks to a generation, depending on the mass (lightest lepton with lightest two quarks etc.). The massive bosons are not part of this classification and all other massive particles are not considered as elementary, but composed of quarks. Our theory instead is based on the 6-dimensional space with its partial geometry only. All particles or constituents (quarks) are "equal" in terms of gaining mass "from scratch" and are the subject of this mechanism. Therefore, it is not surprising that we find a different classification, given by the three levels defined above. These are the particle generations in our theory. But there are similarities to the generations of the SM. The electron clearly belongs to the lightest generation and so differs from the μ and τ. The top quark with its huge mass is clearly part of the third generation, other than the lighter quarks.

At the end of this section we want to understand the role of the non-linearity of our theory for the generation of mass as the central topic of this work. The non-linear term in the fundamental equation (30), (31), respectively, in Eq. (43) for the partial geometry, echoes in the formalism in the terms $b_s(k,l), \underline{b}_s(k,l)$, see Eqs. (91) and (132). These again determine the exponent $\underline{\alpha}_{kl}$ and via relation (47) the number $\underline{n}$, which defines the mass levels discussed above. If $\underline{b}_s(k,l)$ becomes zero, i.e. our theory becomes linear for the considered partial geometry, then $\underline{\alpha}_{kl}, \underline{\lambda}(k,l)$ and $\underline{n}$ are zero as well and thus the functions $\underline{\psi}$ and F^i_{kl} become constants (see (45) and (46)). This means for our mass formulae (70) and (71) with $\underline{n} = 0$ that only the lowest level 0 exists, i.e. the electron mass. The masses of all other particles are "created" by the levels $\underline{n} > 0$ for which the e-function in (70) and (71) becomes > 1. Therefore, the non-linear term in our basic theory creates the mass of all particles heavier than the electron. It provides the "boost" to the higher mass levels $\underline{n} = 2, 4$ which scale in their strength with the power of the inverse of α, the fine structure constant. Mac Gregor's boost is explained this way.

5 Summary and discussion

In this paper we have considered the state of research on the mass energy spectrum of elementary particles and investigated, why phenomenlogical formulae like those of Nambu [10] obviously can reproduce the empirical data well and whether the theory of B. Heim could provide a new approach, leading to more insight into this issue. In a first step we followed the line of thought of Hansson [2], asuming that the mass energy of a particle has a local origin, arising from its self-interactions. From this approach we could already derive a relation for the mass being proportional to the powers of $1/\alpha$ with the fine structure constant α, which in turn agrees with the phenomenological findings of Mac Gregor ([13–18]).

In the main part of this work we adopted the idea that mass might have its origin in a non-linear theory [2] and chose B. Heim's field theory as basis for a concrete investigation. For this purpose we outlined Heim's theory (Section 3) with its basics and properties as a poly-metric structure theory in a 6-dimensional space. To our knowledge it is the first presentation of this theory in English language on this detailed level. As a theory of spacetime, the theory exhibits a natural alignment with the general theory of relativity (GR). Quantisation is introduced by a direct quantisation of actions $\omega = hN$ and with it by a discrete structure field. From this discretisation of the structure field, the field being a radical geometrisation of the phenomenology, follows that spacetime is considered as a medium with a Hilbert functional space in such a manner that an equivalent to the metric structure term of GR arises. Also a smallest two-dimensional area, called metron, is derived by Heim. The resulting mathematic structure of the theory is, similar as Einstein's equation of GR, non-linear. A direct connection to the formalism of quantum mechanics (QM) is only possible in a linear approximation (as outlined in Appendix J) in which the Dirac equation can be derived as a *possible* solution. This is in agreement with the linear nature of QM.

The property of interactions comes into play by Heim's recognition that the obtained mathematical structure suggests a 6-dimensional space with two additional trans-dimensions and thus an overall space with three segments, the known three-dimensional space, time and the two trans-dimensions. From this a poly-geometric structure theory is derived which, in a systematics of physically meaningful combinations of partial metrics, the so-called hermetry forms, provides poly-metric state functions, corresponding to the Christoffel symbols of GR in the macroscopic limit. These functions represent partial geometries which describe deviations from flat Minkowski-like space, called "condensations" by Heim, which correspond to the curvature in 4-dimensional macroscopic space of GR. Different from GR these condensations of partial metric states cannot be transformed away by a coordinate change. They represent forces, i.e. interactions.

In an own new approach with help of the poly-metric state functions and the fundamental relation of the energy momentum tensor to the space-structure-side of the theory, we derive a formula to calculate energy levels which are interpreted as the spectrum of elementary particles (Section 4). Our approach uses both of these ingredients in a straightforward way and differs in this regard from Heim's further development, but bases also explicitly on a quantised space. Whether the assumptions and approximations made on this way are all justified, must be determined by further analysis of the theory, in particular also by calculation of other observables. (From the theory a spin and, more speculative, isospin can be derived, see Appendix K, but they cannot yet be related to the single energy levels.) It then should also be clarified when or whether a derived calculation rule, to use the absolute value of the state functions in the expression for the energy, is to be taken in general.

The non-linearity of the theory generates a formula for the mass-energy which provides a discrete spectrum on several levels. These levels, defined by natural numbers $\underline{n}$, describe a mass hierarchy ranging from the electron on the lowest level, over the muon and tau lepton and nearly all hadrons on the second level and finally the top quark and the heavy bosons on the third level. These levels are not identical to the generations of the SM, but suggest that there is a hierarchy in the mass spectrum which shows a sort of universality, i.e. self-similarity, since the 'basic' pattern, running with a quantum number N, is repeated on different scales (the levels). This is known from non-linear theories.

Our concrete calculations of the mass spectrum show a very good agreement with the experimental masses (average error of 0.43% overall, respectively, 0.85% in a scenario which assigns bosons to even and fermions to odd numbers of "mass units" only) when the lowest mass in this model, the electron mass, is fixed to its empirical value and the second main parameter in the resulting mass-energy formula by a value which corresponds to that of the half inverse of the fine structure constant. The latter fact is in accordance with our inital result mentioned above, including its connection to Hansson's and Mac Gregor's works. The formulae of Nambu, Sidharth and Mac Gregor can be obtained as limits (for bigger N) or special cases of our general result. The results of our 'exact' calculations (calculations E_1 and E_2) are in excellent agreement with the data, especially in the area of the lower masses where the actual error is clearly smaller than the 'statistical' error of our model (see Table 1), which verifies the predictive power of our theory in this domain. (The 'statistical' error results from the fact that single values of the running quantum number N cannot be assigned to certain particles.) The better agreement with the empirical data in this domain, e.g. for the $\mu, \tau, N, \pi^{\pm}$ and ρ, compared to the Nambu formula or to [7, 29], i.e. approaches with somewhat similar resulting formulae, is due to the shape of our result (70) with a slightly more complex structure and dependency on two further parameters which of course also give room for better adjustment to the data.

At this point also the lower limit of the derived mass spectrum has to be discussed. In our model we have assumed that this should be the electron mass and have set the respective parameter to m_e, consistent to the (mainly) phenomenological results of Section 2. Our model, based on Heim's structure theory, has in common with the cited other approaches that it cannot derive the lowest mass from first principles, nor the absolute strength of the boost to the higher mass levels (function f_{ext}, respectively, $1/2\alpha$), since in the absence of natural constants in the fundamental equation (30), the theory does not define an absolute mass scale. Heim, however, succeeds in deriving and calculating a lowest particle mass, which lies very close to that of the electron, by combining results of his structure theory presented here with his phenomenological approach for a corrected large scale behaviour of gravitation (see Appendix M, [43, 44]) in the sense of his principle of equivalence between the geometric structure equation and the phenomenology contained in the energy momentum balance of matter and fields. Thus, in an overall consideration of Heim's work, one could consider the electron mass, gauging the mass scale in our model, as predicted. This of course would require the (here only briefly sketched) phenomenological part of his overall theory to be true.

With the electron having the lightest mass in our model, neutrinos currently cannot be treated in it. In the sense of Hansson's model (Section 2) neutrino masses should correspond to the weak interaction. But a derivation of the weak interaction from Heim's field theory is missing up to now (see below).

On the whole, the value of the presented model and calculation lies in the fact that for the first time the cited phenomenological observations seem to be explained by a more fundamental *non-linear* theory which also offers an approach for a unification of QM, GR and the other interactions – although this theory (by B. Heim) has so far received no attention, let alone recognition. However, a lot of open questions remain and some of them can be connected with some interesting, partly speculative conclusions:

Heim's theory is a holistic theory of spacetime and matter, as matter consists of "condensations" of the discrete spacetime (the metrons). Interactions are implicitly contained, but not easily explicitly derivable.[39] Heim sketched first approaches for (microscopic) gravity and EM (see Appendix G and Appendix J for EM in the linear limit), we made a proposal how the gauge symmetries of the SM ($U(1)SU(2)SU(3)$) could be explained by the 6-dimensional Heim space (in Appendix C) and drafted a qualitative ansatz for the r-dependent course of the strong interaction (Appendix L), but a concrete formalism for deriving the three gauge interactions from the hermetry forms and the condenser classes is still missing. Such an approach should reproduce the known laws of these interactions in respective limits. Thereby, the nature of the "internal" degrees of freedom like (weak) isospin, flavour and colour would have to be clarified, too. The absolute strength of the interactions, i.e. the coupling constants will most likely not be derivable from the theory, as the pure geometric structure theory may not accomplish this. (In GR the empirical "anchoring" happens in the form of the gravitational constant as proportionality factor in front of the energy momentum tensor.)

In our theory we have learned that the exponent $\underline{\alpha}$ determines the transition from the composition field to the field of the particular partial geometry and its condensation and thereby sets a measure of the respective interaction. According to our formula (47) $\underline{\alpha}$ in turn is determined by the (mass) level parameter $\underline{n}$ and by the function f which obviously is crucial for the course of the strong interaction in our model, showing that these two quantities describe the interaction and also the values of the particle masses (via Eq. (70)).

When in our model, as stated above, the boost to the higher mass levels is determined by the inverse of the fine structure constant, as difference in strength between EM and the strong interaction, then also the masses of the non-hadron particles, the μ, τ and the heavy gauge bosons are generated by this "force". Why don't they "feel" the strong interaction? This issue may once again indicate that a clear derivation of the interactions from Heim's theory is still missing, but also more concretely that the hermetry forms c and d of the theory should include more than only the strong interaction.

Can an equivalence be found between Heim's theory and Quantum Field Theory (QFT)? This question will be strongly related to the issue of the interactions and the way how the linear limit to quantum theory is derived. It should be emphasised that QFT is non-linear as well, perturbation theory and Feynman graphs only hide this property.

If quantum theory, in form of the Dirac equation, can be derived as a possible linear limit from Heim's theory (Appendix J), can the measurement problem of QM be solved by such a non-linear basic theory? Several attempts have been made over the decades to solve the issue ([61–69]), non-linear approaches played an important role (e.g. the Newton–Schrödinger equation, [61, 62]). Hansson ([69]) suggested that the non-abelian (non-linear) gauge theories of the SM could be the source of the collapse of linear QM not only in the measurement process, but in general, causing different coherence lengths, e.g. between photons, underlying only the abelian QED, and electrons, being also subject to the non-linear weak interaction. Heim's theory and our model are basically non-linear and therefore provide such a mechanism to "destroy" coherent linear superpositions of quantum states. To derive concrete higher-order expressions from the non-linear basic equation (30), which add to the pure linear Dirac terms of Appendix J, could be a next interesting step of research in this domain.[40]

In this context, considering the border between the quantum and the classical regime, it should be

39 Except for gravity on the macroscopic scale, since Heim's basic equation merges into Einstein's field equation in the macroscopic limit, see subsection 3.3.

40 Note that the non-linear term of (30) should carry the same pre-factor in terms of the imaginary unit i as the first term on the r.h.s. of Eq. (170) which contains the partial derivations of the 3 space coordinates, provided that both terms would not be changed in their "phase" in the course of the transformation towards a Dirac-like equation, as outlined in Appendix J. As a consequence of the structure of the Dirac equation, the non-linear term then would carry the same factor in terms of i as the time derivation in (170), different from the non-relativistic Newton–Schrödinger equation. This should lead to damping, similar to master equations with a decoherence term [70].

noted that in Heim's theory a quantisation of GR by means of the canonical formalism is not necessary, nor appropriate. Heim's fundamental equation instead defines the quantisation in his theory and turns into the Einstein equation in the macroscopic realm (see (17) and (18)).

Finally, we remind of the fact that rest mass obviously can be measured without a limit of accuracy, there is no explicitly known uncertainty principle for it. We conclude from this work, in agreement with Heim ([43] page 40), that the reason seems to be its non-linear origin.

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Appendix A: Non-hermitian spacetime structure in the R_4

We summarise the content of chapter I.3, pages 26–28, in [43]:

We consider as a general case $m \geq 1$ non gauge-invariant and $n-m$ $(n>m)$ gauge-invariant interaction fields. For each field a characteristic partial event structure determined by the field symmetries can be defined, given through a geodesic coordinate system in spacetime $\vec{\xi}_p^{(j)} = \vec{\epsilon}_p^{(j)} \xi_p^{(j)}$ with $1 \leq j \leq n$ different interactions and $1 \leq p \leq 4$ coordinates. The unit vectors $\vec{\epsilon}_p^{(j)}$ do not have to be necessarily an orthogonal system. The 4 geodesic coordinates can only be related to the 4 Cartesian coordinates x_k of the empty R_4 so that transformations $\xi_p^{(j)}(x_1, x_2, x_3, x_4)$ exist which must be those of the global Poincare group.

Regarding the structural properties in the R_4 the n fields can be split into the subsets of the m non-gauge-invariant and $n-m$ gauge-invariant interactions. The total differentials per coordinate for both subsets can be written as $\mathrm{d}\vec{z}_p^{+} = \sum_{j=1}^{m} \mathrm{d}\vec{\xi}_p^{(j)}$ and $\mathrm{d}\vec{z}_p^{-} = \sum_{j=m+1}^{n} \mathrm{d}\vec{\xi}_p^{(j)}$. So, vectorial line elements appear $\mathrm{d}\vec{s}_{\pm} = \sum_{p=1}^{4} \mathrm{d}\vec{z}_p^{\pm}$ with $\vec{z}_p^{\pm} = \vec{\epsilon}_p^{\pm} z_p^{\pm}$.

The vectorial line element in the R_4 then is $\mathrm{d}\vec{s} = \mathrm{d}\vec{s}_+ + \mathrm{d}\vec{s}_-$ where in general $\cos(\mathrm{d}\vec{s}_+, \mathrm{d}\vec{s}_-) \neq 0$ holds.

Thereby the metric for $\vec{z}^{\pm} = (\vec{z}^{\pm})^*$ in the R_4 becomes $\mathrm{d}s^2 = \mathrm{d}s_+^2 + 2\mathrm{d}\vec{s}_+\mathrm{d}\vec{s}_- + \mathrm{d}s_-^2$, or with the definition of total differentials $\mathrm{d}z_p^{\pm} = \frac{\partial z_p^{\pm}}{\partial x^i}\mathrm{d}x^i$

$$\begin{aligned} \mathrm{d}s_+^2 &= \sum_{p,q=1}^{4} \vec{\epsilon}_p^{\,+}\vec{\epsilon}_q^{\,+} \frac{\partial z_p^+}{\partial x^i}\frac{\partial z_q^+}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k = g_{ik}^{(1)}\mathrm{d}x^i\mathrm{d}x^k, \\ 2\mathrm{d}\vec{s}_+\mathrm{d}\vec{s}_- &= 2\sum_{p,q=1}^{4} \vec{\epsilon}_p^{\,+}\vec{\epsilon}_q^{\,-} \frac{\partial z_p^+}{\partial x^i}\frac{\partial z_q^-}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k = g_{ik}^{(2)}\mathrm{d}x^i\mathrm{d}x^k, \\ \mathrm{d}s_-^2 &= \sum_{p,q=1}^{4} \vec{\epsilon}_p^{\,-}\vec{\epsilon}_q^{\,-} \frac{\partial z_p^-}{\partial x^i}\frac{\partial z_q^-}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k = g_{ik}^{(3)}\mathrm{d}x^i\mathrm{d}x^k \\ \text{or}\quad \mathrm{d}s^2 &= (g_{ik}^{(1)} + g_{ik}^{(2)} + g_{ik}^{(3)})\mathrm{d}x^i\mathrm{d}x^k = g_{ik}\mathrm{d}x^i\mathrm{d}x^k. \end{aligned} \tag{72}$$

The coefficients of this homogeneous quadratic differential form $g_{ik}^{(\beta)}$ or g_{ik} are tensor components, namely in relation to the invariance against the global Poincare group. They are field functions of the x_k of an empty R_4. $g_{ik}(x_1, x_2, x_3, x_4)$ is the field of the fundamental metric tensor which designates the allowed event structures of the R_4 in an invariant form which correspond to that material structure phenomenologically represented by the $T_{ik} \neq T_{ki}$.

The symmetry of $g_{ik}^{(1)} = g_{ki}^{(1)}$ and $g_{ik}^{(3)} = g_{ki}^{(3)}$ is immediately evident, while $g_{ik}^{(2)} \neq g_{ki}^{(2)}$ proves to be asymmetric so that $g_{ik} \neq g_{ki}$ overall is asymmetric as well.

So, in the R_4 a non-hermitian fundamental metric tensor of a general Cartan geometry holds as a consequence of a geometrisation of general interaction fields of matter. Because of their non-hermiticity and being tensors, T_{ik} and g_{ik} each can be split into a hermitan ('+') and anti-hermitian ('−') part, $T_{ik} = T_{ik}^+ + T_{ik}^-$, $g_{ik} = g_{ik}^+ + g_{ik}^-$ where the index $\pm$ of course no longer relates to the $z_p^{\pm}$, but to the sign of the hermitian conjugation.[41]

Appendix B: Derivation of equation (17) according to Heim [43]

Subsequently we reproduce the content of pages 37–39 of chapter II.1 in [43]:

As stated in Subsection 3.3, from the discretisation of the structure field follows that the spacetime R_4 must be considered as a medium with a Hilbert functional space, i.e. a convergent state function (field) ϕ_{km}^i of the metric state of spacetime must exist and a hermitian state operator C_p acting on ϕ_{km}^i in such a manner that an equivalent to the metric structure term arises $C_p\phi_{km}^i \to C_p\Gamma_{km}^i = R_{kmp}^i$. This finding shall be derived hereinafter.

41 In general, a tensor can be split into a hermitan ('+') and anti-hermitian ('−') part by $A = \frac{1}{2}((A + A^{\times}) + (A - A^{\times})) = \frac{1}{2}(A^+ + A^-)$ with $A^{\times}$ being the adjoint of A. From the defintion of A^+ and A^- follows that $(A^+)^{\times} = A^+$ is hermitian and $(A^-)^{\times} = -A^-$ anti-hermitian.

While the Γ^i_{km} do not have to be convergent, for the ϕ^i_{km} convergence can be achieved so that a spacetime integral of the form is $J^i_{km} = \int_\Omega \phi^i_{km}\phi^{i*}_{mk}\mathrm{d}\Omega < \infty$, which allows the normalisation $J^i_{km} = 1$. But this convergence in general gets lost in the transition $\phi^i_{km} \to \Gamma^i_{km}$ into the macroscopic continuum. So, the convergent ϕ^i_{km} describe the metric but non-hermitian state of the R_4 in the microscopic realm and continue according to the principle of correspondence as not necessarily convergent Γ^i_{km} into the macroscopic continuum of the R_4. Therefore, in the microscopic R_4 there must be $1 \le p \le 4$ hermitian functional operators C_p which separately act on the ϕ^i_{km} so that for each index p holds

$$\int_\Omega ((\phi^{(p)}_{km})^\times C_{(p)}\phi^{(p)}_{km} - \phi^{(p)}_{km}(C_{(p)}\phi^{(p)}_{km})^\times)\mathrm{d}\Omega = 0. \tag{73}$$

The functional operators act on the ϕ^p_{km} as $C_p\phi^p_{km}$ so that in the transition into the macroscopic continuum, beside $\phi^i_{km} \to \Gamma^i_{km}$, the action of the operators gives $C_p\phi^p_{km} \to R_{km}$, the metric structure tensor. This transition means at the same time that the discontinuity η_{ik} from (15) goes over to the steady function $w\eta_{ik} \to T_{ik} - \frac{1}{2}g_{ik}T$ (up to a proportionality factor). Then $R_{ik} \sim w\eta_{ik}$ follows, as $R_{ik} = \alpha(T_{ik} - \frac{1}{2}g_{ik}T) = \alpha W_{ik}$. On the other hand $C_p\phi^p_{km} \to R_{km}$ holds, as stated above, so $C_p\phi^p_{km} \to \alpha W_{km}$ can be set.

Now αW_{km} can be constituted as a sum of 4 parts (correspondent to the summation over p) $\alpha W_{km} = \sum_{j=1}^4 G_{(j)km}$ which differ in their type (j). Ordered in a way so that $p = j$, we get $C_{(p)}\phi^{(p)}_{km} \to G_{(p)km}$. Due to the quantum principle c. the $G_{(p)km}$ can be described by state operators of a state space (as a subspace of the abstract function space) in the same way as the ϕ^i_{km} whose state functions ψ can be interpreted as probabilities (of future possibilities). The respective hermitian linear operators of this state space be $H^{(p)}_{km}$ and $L^{(p)}_{km}$ with the eigenvalues $h^{(p)}_{km} = h^{(p)*}_{km}$ and $l^{(p)}_{km} = l^{(p)*}_{km}$ and the state function ψ, for which the normalisation $\int \psi\psi^*\mathrm{d}\Omega = 1$ holds in terms of the probability interpretation of quantum theory. Therefore, when transferring to the microscopic realm, $G_{(p)km}\psi \to H^{(p)}_{km}\psi = h^{(p)}_{km}\psi$ and $\phi^{(p)}_{km}\psi \to L^{(p)}_{km}\psi = l^{(p)}_{km}\psi$ is to be set.

In contrast, in the macroscopic area $\ddot{x}^i = -\Gamma^i_{km}\dot{x}^k\dot{x}^m$ holds and due to $\Gamma^i_{km} \to \phi^i_{km}$ also $\ddot{x}^i \to -\phi^i_{km}\dot{x}^k\dot{x}^m$. Integrals of the form $\int \ddot{x}^i\mathrm{d}x^i$ are always proportional to energies which holds microscopically, too. Also, a metric structuring of the R_4 will require energetic effort so that Heim heuristically conceives $H^{(p)}_{km} \sim L^{(p)}_{km}$ in terms of $H^{(p)}_{km} = \lambda_{(p)}(k,m)L^{(p)}_{km}$. With

$$H^{(p)}_{km}\psi = \lambda_{(p)}(k,m)L^{(p)}_{km}\psi = \lambda_{(p)}(k,m)l^{(p)}_{km}\psi \quad \text{one gets}$$

$$\begin{aligned} 0 &= \int (\psi^* H^{(p)}_{km}\psi - \psi(H^{(p)}_{km}\psi)^*)\mathrm{d}\Omega \\ &= (\lambda_{(p)}(k,m)l^{(p)}_{km} - (\lambda_{(p)}(k,m)l^{(p)}_{km})^*)\int \psi\psi^*\mathrm{d}\Omega \end{aligned} \tag{74}$$

which because of $\int \psi\psi^*\mathrm{d}\Omega = 1$ and $l^{(p)}_{km} = l^{(p)*}_{km}$ can only be fulfilled by $\lambda_{(p)}(k,m) = (\lambda_{(p)}(k,m))^*$. So, the λ_p obviously have the property of eigenvalues, too. For the transition of the $G_{(p)km}$ into the microscopic realm therefore holds

$$\begin{aligned} &G_{(p)km}\psi \to H^{(p)}_{km}\psi = \lambda_{(p)}(k,m)L^{(p)}_{km}\psi = \lambda_{(p)}(k,m)\phi^{(p)}_{km}\psi, \\ &\text{thus} \quad G_{(p)km} \to \lambda_{(p)}(k,m)\phi^{(p)}_{km}. \end{aligned} \tag{75}$$

On the other hand the transition $G_{(p)km} \to C_{(p)}\phi^{(p)}_{km}$ holds (see above), which provides $C_{(p)}\phi^{(p)}_{km} = \lambda_{(p)}(k,m)\phi^{(p)}_{km}$, i.e. Eq. (17).

In this system, the $\lambda_p = \lambda^*_p \neq 0$ are eigenvalues which build discrete point spectra. The indexing (k,m) at the eigenvalue symbol $\lambda_p(k,m)$ refers to the associated structure (state) function ϕ^p_{km}. So, $C_{(p)}\phi^{(p)}_{km} = \lambda_{(p)}(k,m)\phi^{(p)}_{km}$ indeed fulfils the empirical quantum principle c., but, as stated in Subsection 3.3, the ϕ^p_{km} cannot be interpreted as probability functions, since the operators C_p act (due to $C_p\phi^p_{km} \to R_{km}$) as functional operators in terms of non-linear connections so that the solutions (as metric structure functions of the R_4) do not superpose additively, which is the precondition for the probability interpretation in quantum mechanics.

Appendix C: Dimensions and symmetries

C.1 Heim's dimension formula

Here we summarise the results of [43], chapter II.1, pages 45–46 and 48:

Heim derives his dimension formula in the following way: From relation (20) for the eigenvalues $\lambda_m(k,m) = \lambda_m(m,k) = 0$ follows that $2n^2$ values are zero, n being the dimension of the considered space wherein n values are double counted. So $2n^2 - n$ different are zero which have to be subtracted from the n^3 eigenvalues $\lambda_p(k,m)$ in total. Thus the number of non-empty values amounts to $z = n^3 - 2n^2 + n = n(n^2 - 2n + 1) = n(n-1)^2$.

Next the further diminution of the non-zero eigenvalues by the property $\lambda_p(m,m) = 0$, described in footnote 13 of Subsection 3.3, is taken into account, which means a reduction of $n^2 - n$ (no redundant counting of the $\lambda_m(m, m)$) so that the overall number of non-zero values becomes

$$r = z - (n^2 - n) = n(n-1)^2 - n^2 + n = n(n-1)(n-2) = 6\binom{n}{3}. \quad (76)$$

To find a mapping $R_n \to R_N$ with $N \geq n \geq 0$, r must be equaled to a quantity $N^2 - \alpha N$, where $\alpha = \text{const} > 1$ is a still variable factor. For the resulting quadratic equation $N^2 - \alpha N = r$ one gets the solutions $N_{1,2} = \alpha/2 \pm \sqrt{r + \alpha^2/4}$ and can set $\alpha = 2$ to ensure a whole-number result for N. Thus

$$N = 1 \pm \sqrt{r+1}, \quad r = n(n-1)(n-2), \quad 0 \leq n \leq N. \quad (77)$$

In this dimensional law the number $r + 1$ must be a square number in order to obtain an integer N. For the R_4, i.e. $n = 4$, we get $r + 1 = 25$ and for $n = 6$, $r + 1 = 121$ which provides $N(n = 4) = 6$ and $N(n = 6) = 12$. Therefore, the R_4, according to $R_4 \to R_6 \to R_{12}$, can only be the subspace of a R_6 or R_{12} while all other possibilities must be excluded. Heim first of all chose and analysed the lower dimensional case, the R_6, and founded his theory, as elaborated in [43, 44], on it.

C.2 Later expansion of the model space

Later Heim and W. Dröscher also considered the R_{12} space and came to the conclusion that the additional 6 coordinates $x_7 - x_{12}$, which do not contain or transport energy (this is restricted to the first 6 coordinates $x_1 - x_6$), can be grouped into two further subspaces, I_2 of the coordinates x_7 and x_8 and G_4, containing the coordinates $x_9 - x_{12}$. In a longer analysis, published in an additional chapter IV 5. “Background and Sources of the Quantum Principle” (translation by the author) in [43], they concluded that the I_2 is to be considered as subspace of two dimensions containing information. The remaining subspace G_4 can only be assigned with a length, no more can be said about it. In their concept, the I_2 subspace causes that observables in the R_4 can only detected with uncertainty and probabilities. They speculate that the quantum theory could be interpreted in a way that its sources lie in the G_4 with timeless structures which become effective via the mapping chain

$$G_4(x_9, \ldots, x_{12}) \to I_2(x_7, x_8) \to (x_5, x_6) \to x_4 \to R_3 \quad (78)$$

so that quantum structures, changing in time, manifest in the physical R_3 space of the universe. This highly abstract model is based on formal and physical considerations about the dimensions space, including the cosmological part of Heim’s theory, and has been continued in Heim’s third book [71] (with the assistance of W. Dröscher).

In the eyes of the author it is not *a priori* clear that the extended model of the R_{12} or the R_8 ($R_6 + I_2(x_7, x_8)$), which Dröscher/Häuser use in their ‘Extended Heim Theory’ ([45–49]), is better capable to explain the existence of the quantum world than the R_6. As stated in Subsection 3.3, already the R_6 (via the trans-coordinates x_5, x_6) leads to a world with an open future and together with the existence of binary alternatives to the abstract quantum theory (see [54] and the further arguments in C.3).

Another question is whether the ‘Extended Heim Theory’ will be able to provide more specific answers with regard to the derivation of the known interactions. With the 8-dimensional EHT Dröscher/Häuser presented an approach which provides 12 [45], respectively, 15 [47] hermetry forms and thus a much bigger set of partial geometries than the original Heim theory. They assign the combinations of 4 subspaces (the three $\kappa_{(i)}$ of Subsection 3.4.3 plus a fourth subspace ($\kappa_{(4)}$) of the coordinates x_7, x_8) to matter particles, respectively, interaction bosons and suggest that the transformation from the partial geometries to Lagrangians could be accomplished by the non-linear sigma model [72]. However, as far as we know, they have not yet published concrete derivations on this basis, but refer to the known Lagrangians of the gauge theories of the SM [49]. To map to the known forces still seems to be easiest for the electromagnetic interaction. Here the metric can be well approximated by $g_{ik} = g_{ik}^{(0)} + h_{ik}$, where $g_{ik}^{(0)}$ is the flat metric of the Euclidean 4-dimensional space and the h_{ik} are small quantities whose products are negligible. The derivation in [46] gives a result containing the expected terms with the vector and scalar potential of EM, but also an additional tensor potential which is not present in classical electrodynamics. It remains unclear, how a mapping can be derived for the strong interaction where a similar approximation does not seem to be reasonable. Currently neither Heim’s theory nor the EHT can present an elaborated mapping to the weak and the strong interaction.

C.3 Heim space and gauge symmetries

(*Consideration of the author, not by Heim*)

The existence of binary alternatives can be modelled mathematically easiest with the $SU(2)$ group. The relevance of this group and symmetry in the physical R_4 can immediately be seen from the symmetry of the Lorentz group, valid in the Minkowskian R_4, which is locally isomorphic to a $SU(2) \otimes SU(2)$ space with 3 generators of spatial rotations and 3 of Lorentz boosts.[42]

In the 6-dimensional case of the Heim space the symmetry of the R_6, defined by (22), should be locally isomorphic to $SU(2) \otimes SU(2) \otimes U(1) \otimes U(1)$ if the dimensions x_5 and x_6 are distinguishable from each other (which they are in Heim's approach) and follow a $U(1)$ symmetry, i.e. can be rotated in the complex plane, but without a change of the length (radius).

We compare this overall symmetry to that of the $SU(3)$ which is an eight-dimensional compact group. A parameterisation of the $SU(3)$ can be obtained which matches a duplicated structure of the $SU(2) \otimes U(1)$ groups, see [73, 74]. Such a structure is called Cartan decomposition:

$$D^{(3)}(\alpha, \beta, \gamma, \theta, a, b, c, \phi) = D^{(2)}(\alpha, \beta, \gamma) e^{i\lambda_5 \theta} D^{(2)} \times (a, b, c) e^{i\lambda_8 \phi} \tag{79}$$

$D^{(3)}$ denotes an arbitrary element of $SU(3)$, and $D^{(2)}$ is an arbitrary element of $SU(2)$ as a subset of $SU(3)$. The λ_i are 3×3-matrices, therefore also $D^{(3)}$ and $D^{(2)}$ are 3×3-matrices. The two exponential functions $e^{i\lambda_5 \theta}$ and $e^{i\lambda_8 \phi}$ describe the two $U(1)$ manifolds as subgroup and coset in $SU(3)$ [73].

Thus, we have shown the local isomorphism of the 6-dimensional Heim space to a duplicated structure of the $SU(2) \otimes U(1)$ and, via these, to the $SU(3)$ if the "trans"-dimensions x_5, x_6 follow a $U(1)$ symmetry. This finding can be used to conclude that the 6-dimensional Heim space carries a symmetry which contains a $SU(2) \otimes U(1)$ as a subset, i.e. the gauge symmetry of the electro-weak interaction, and furthermore the $SU(3)$, the gauge symmetry of the strong interaction (according to the QCD).

In a similar way, Görnitz and Schomäcker deduced the gauge groups $U(1), SU(2)$ and $SU(3)$ as structures of these interactions [73]. But their starting point was the abstract quantum space which, according to [54, 73] (and citations therein), carries a $U(1) \otimes SU(2)$ symmetry. The electro-weak interaction, acting on real particles/quanta, directly can be derived from this. Concerning the strong interaction, they argue [73] that it only acts on "virtual" quanta, the quarks and gluons, never detected as free particles/quanta, and draw the analogy of the transition from real to complex functions in quantum theory, a "duplication" of real functions, to conclude that a duplication of the $U(1) \otimes SU(2)$ structure is necessary to describe these "virtual" quanta and their interaction, providing the $SU(3)$ (in an analogous way as above).

This derivation now suggests a further connection, viz. that of the 6-dimensional Heim space to the space of abstract quantum theory. A duplication of the symmetry group of the quantum space seems to lead to Heim space – which could mean that the 6-dimensional Heim space includes quantum theory and that it doubles already its space and so accounts for interaction which always "implies a division into separate spaces, . . . since ultimately the term 'interaction' is useful only for things separated from each other" (to quote [73]).

Appendix D: Further contractions of R^i_{kmp}

We recap the content of pages 66–67 in chapter II.3 of [43]:

Beside the well-known contractions of the Riemannian curvature tensor, the Ricci tensor $R_{km} = R^p_{kmp}$ and the curvature scalar $R = R^k_k$, there is a further possibility to build a trace with $i = k$,

$$R^k_{kmp} = \Gamma^k_{kp,m} - \Gamma^k_{km,p} + \Gamma^k_{ms}\Gamma^s_{kp} - \Gamma^k_{ps}\Gamma^s_{km} := B_{mp} = -B_{pm} \tag{80}$$

with $B_{mm} = 0$. Refering to Eqs. (26)–(28), this antisymmetric tensor arises from the trace

$$(C\ \mathrm{Tr}\,\hat{\{\}})_{mp} \rightarrow B_{mp} \quad \text{or} \quad (\vec{\lambda} \times \mathrm{Tr}\,\hat{\{\}})_{mp} \rightarrow B_{mp}. \tag{81}$$

Because of $B_{mm} = 0$ and $B_{mp} \rightarrow \lambda_p(k, m) \left\{ {k \atop km} \right\}$ also $\lambda_m(k, m) \left\{ {k \atop km} \right\} = 0$ holds and thus $\mathrm{Tr}\,(\vec{\lambda} \times \mathrm{Tr}\,\hat{\{\}}) = \vec{\lambda}\ \mathrm{Tr}\,\hat{\{\}} = 0$ in Heim's compact notation. In the hermitian R_6 this can generally only be met by $\cos(\vec{\lambda}, \mathrm{Tr}\,\hat{\{\}}) = 0$, i.e. by the orthogonality $\vec{\lambda} \perp \mathrm{Tr}\,\hat{\{\}}$. Due to the transition $\mathrm{Tr}\,\hat{\{\}}_m = \left\{ {k \atop km} \right\} \rightarrow \Gamma^k_{km}$ in the macroscopic realm this orthogonality can be further interpreted. With the theorem $\left\{ {k \atop km} \right\} = \frac{1}{2} \frac{\partial}{\partial x^m} \ln g = \frac{\partial}{\partial x^m} \ln w$ (see [52]), g being the determinant of the metric tensor in the R_6 and $w = \sqrt{-g}$ (compare Subsection 3.3), we can write

$$\vec{\lambda} \perp \mathrm{Tr}\,\hat{\{\}} \rightarrow \nabla_6 \ln w. \tag{82}$$

This relation states that the $\vec{\lambda}$ are always perpendicular to the gradient of the logarithm of the functional determinant w in the R_6.

Heim additionally considers the contraction R^m_{kmp} of the Riemannian curvature tensor, yielding the negative Ricci tensor $R^m_{kmp} = -R_{kp}$. With this contraction another

42 Remember that Varlamov bases his approach for a particle mass spectrum on the Lorentz group symmetry of spacetime [7].

transition, beside the one of Eq. (26) et sqq., $C_p \left\{ {p \atop km} \right\} = \lambda_p(k,m) \left\{ {p \atop km} \right\} \to R_{km}$, can be achieved:

$$C_p \left\{ {m \atop km} \right\} = \lambda_p(k,m) \left\{ {m \atop km} \right\} \to -R_{kp} \tag{83}$$

Heim speculates that this further relation could have a physical meaning, too, and interrelates it to a virtual spectrum of discrete structure levels with a curvature measure (condensation) of opposite sign, instead of the real spectrum, represented by (26) et sqq., with a positive curvature measure (condensation). If this corresponded to the microscopic properties of the R_6, then real and virtual states of structure levels could interchange as temporal oscillations and establish a dynamic equilibrium of per se non-stationary microscopic structure fluxes (of condensation) in case of stability. Heim relates this possible picture to the perpetual exchange mechanism, which he calls condensation flux, referred to in Subsection 3.5.3. He further speculates whether notions like vacuum polarisation, virtual terms and the empirical Casimir effect could be interpreted by these fluctuations in the microscopic realm.

Appendix E: Structures and metric tensors

E.1 Partial structures from the phenomenology of the R_4

An analysis by Heim in [43], pages 69–74, of the structure of the energy momentum density tensor T_{ik} in the R_6 will give us some insight how T_{ik} can be composed of the empirical physical fields in the *macroscopic* realm:

The canonical energy density tensors in the R_4 can always be written as iterations of field tensors. So, it is obvious to also set T_{ik} as the iteration of a unified field tensor M_{ik} in the R_6 in the macroscopic realm, $T_{ik} = \sum_{\nu=1}^{6} M_{i\nu} M_{\nu k}$, and to determine the M_{ik} from the phenomenologically known T_{ik}. As $T_{ik} = T_{ki}^*$, it can only be $M_{ik} = \pm M_{ki}^*$. In the empirical domain of EM (d1) M is given by the field tensor $F(\vec{E},\vec{H}) = -F^\times$ with $F_{mm} = 0$ in the R_4. On the other hand, a photon (as gauge particle of EM), although with zero rest mass and imponderable, must be attached with a field mass through its energy. Therefore, the spatial gravitational vector fields $\vec{G}$ and $\vec{\mu}$ must exist for a photon, too, and in a form that the spacetime segment of M, $M_{(4)} \to F$ for $\vec{G} \to \vec{0}$ and $\vec{\mu} \to \vec{0}$. From this, it could be concluded that clearly $M = -M^\times$ holds with $M_{mm} = 0$. The $\binom{6}{2} = 15$ components of the uniform field tensor M of a photon in the R_6 are linearly composed of the spatial vector fields $\vec{E}, \vec{H}, \vec{G}$ and $\vec{\mu}$, according to the empirical principles d1 and d2. Here, the field energy=mass of the EM field is to be considered as the source of $\vec{G}$ and $\vec{\mu}$. But field source and field are always a unit so that an additional field $\vec{K}$ must be conceived which accounts for the coupling between the EM and the gravitational field structure. So, M is composed of 5 spatial vector fields whose 3×5 components can fill the 15 components of the R_6 tensor. These 5 phenomenological spatial fields which define the $M_{ik} = -M_{ki}^*$ with $M_{kk} = 0$ build three phenomenological groups of phenomena, namely $(\vec{E},\vec{H})$, $(\vec{G},\vec{\mu})$ and $\vec{K}$.[43]

E.2 Composition metric tensors per Hermetry

Referring to chapter VI.1 of [44], pages 80–81, we state the concrete form of the $g^{(x)}$ which correlate the partial metric structures per Hermetry, and start with the form d in which all partial structures are active, i.e. "condensed":

$$g^{(d)} = \begin{pmatrix} g^{(11)} & g^{(12)} & g^{(13)} \\ g^{(21)} & g^{(22)} & g^{(23)} \\ g^{(31)} & g^{(32)} & g^{(33)} \end{pmatrix} \tag{84}$$

The columns and rows represent the structure units indexed with μ, ν in (38). On this "super" tensor so-called sieve operators $S[n]$ can act which make the structure units in (84) Euclidian:

$$g^{(c)} = S(2)g = \begin{pmatrix} g^{(11)} & \kappa^{(1)} & g^{(13)} \\ \kappa^{(1)} & E & \kappa^{(3)} \\ g^{(31)} & \kappa^{(3)} & g^{(33)} \end{pmatrix} \tag{85}$$

$$g^{(b)} = S(3)g = \begin{pmatrix} g^{(11)} & g^{(12)} & \kappa^{(1)} \\ g^{(21)} & g^{(22)} & \kappa^{(2)} \\ \kappa^{(1)} & \kappa^{(2)} & E \end{pmatrix} \tag{86}$$

$$g^{(a)} = S(2,3)g = \begin{pmatrix} g^{(11)} & \kappa^{(1)} & \kappa^{(1)} \\ \kappa^{(1)} & E & E \\ \kappa^{(1)} & E & E \end{pmatrix} \tag{87}$$

43 The speculative postulation of an additional new field is currently not backed by direct empirical hints, but otherwise its existence cannot be excluded *a priori*. In the context of a 6-dimensional theory it should not be astonishing that so far unknown forces (fields) come into play. The gravitational Meso field $\vec{\mu}$ was already the first "new physics" conceived by Heim in the frame of his theory.

Appendix F: Solution of the hermetric fundamental equation (30) and further relations

We refer to the calculations of chapter IV.2, pages 196–205 and 211 in [43], but apply directly normal differentiation and integration which is valid for Heim's third area of validity, considered here.

F.1 Solution of equation (30)

Based on the relations for the coefficients a_{lm} of subsection 3.4.4 and on $\hat{\{\}} = \hat{\{\}}^{\times}$ being hermitian, it directly follows $\left\{ {i \atop ml} \right\} = \left\{ {i \atop lm} \right\}$ and thus the system of symmetry relations $a_{ml}\left\{ {i \atop mm} \right\} - a_{lm}\left\{ {i \atop ll} \right\} = 0$ or as proportionality $\left\{ {i \atop mm} \right\} = \frac{a_{lm}}{a_{ml}}\left\{ {i \atop ll} \right\}$. So we have two systems of proportionalities at hand, the last one and $\left\{ {i \atop ml} \right\} = a_{ml}\left\{ {i \atop mm} \right\}$, already introduced above in the cited subsection. We can use them to substitute in the general equation (30) so that uniformly only the covariant components k, l are related. The mixed-variant sum over s in the quadratic term runs in $1 \le s \le q$, as only in this interval zero factors $\left\{ {i \atop \bar{s}p} \right\} = 0$ do not occur. So, we get

$$\left\{ {i \atop km} \right\} = a_{km}\left\{ {i \atop kk} \right\} = \frac{a_{km}}{a_{kl}}\left\{ {i \atop kl} \right\} \quad \text{and}$$
$$\left\{ {i \atop ls} \right\} = a_{ls}\left\{ {i \atop ll} \right\} = \frac{a_{ls}}{a_{lk}}\left\{ {i \atop lk} \right\} = \frac{a_{ls}}{a_{lk}}\left\{ {i \atop kl} \right\}, \quad \text{thus}$$
$$\left\{ {i \atop ls} \right\}\left\{ {s \atop km} \right\} = \frac{a_{ls}}{a_{lk}}\frac{a_{km}}{a_{kl}}\left\{ {i \atop kl} \right\}\left\{ {s \atop kl} \right\}. \quad \text{Similarly follows}$$
$$\left\{ {i \atop ms} \right\} = a_{ms}\left\{ {i \atop mm} \right\} = \frac{a_{ms}}{a_{mk}}\left\{ {i \atop km} \right\} = \frac{a_{ms}}{a_{mk}}\frac{a_{km}}{a_{kl}}\left\{ {i \atop kl} \right\}. \tag{88}$$

Inserting into (30) gives

$$\left(\frac{a_{km}}{a_{kl}}\partial_l - \partial_m\right)\left\{ {i \atop kl} \right\} + \left(\frac{a_{ls}}{a_{lk}}\frac{a_{km}}{a_{kl}} - \frac{a_{ms}}{a_{mk}}\frac{a_{km}}{a_{kl}}\right) \times \left\{ {i \atop kl} \right\}\left\{ {s \atop kl} \right\} = \lambda_m(k,l)\left\{ {i \atop kl} \right\}. \tag{89}$$

With the abbreviations $\frac{a_{km}}{a_{kl}} = a_m(k,l)$ and $\frac{a_{ls}}{a_{lk}}\frac{a_{km}}{a_{kl}} - \frac{a_{ms}}{a_{mk}}\frac{a_{km}}{a_{kl}} = b_{ms}(k,l)$ Eq. (89) reads

$$(a_m(k,l)\partial_l - \partial_m)\left\{ {i \atop kl} \right\} + b_{ms}(k,l)\left\{ {i \atop kl} \right\}\left\{ {s \atop kl} \right\} = \lambda_m(k,l)\left\{ {i \atop kl} \right\}. \tag{90}$$

If we now sum over the hermetric index m and use the further abbreviations $a(k,l) = \sum_{m=1}^{q} a_m(k,l)$, $b_s(k,l) = \sum_{m=1}^{q} b_{ms}(k,l)$ and $\lambda(k,l) = \sum_{m=1}^{q}\lambda_m(k,l)$, then we get with $a(k,l)\partial_l - \sum_{m=1}^{q}\partial_m = (a(k,l)-1)\partial_l - \sum_{m\neq l}\partial_m$

$$\left((a(k,l)-1)\partial_l - \sum_{m\neq l}\partial_m\right)\left\{ {i \atop kl} \right\} + b_s(k,l)\left\{ {i \atop kl} \right\}\left\{ {s \atop kl} \right\} = \lambda(k,l)\left\{ {i \atop kl} \right\}. \tag{91}$$

This equation can be multiplied with the coefficient $b_i(k,l)$ and summed over $1 \le i \le q$, leading to covariant selectors (tensors) $\phi_{kl} = b_i(k,l)\left\{ {i \atop kl} \right\}$. As further $b_i(k,l)b_s(k,l)\left\{ {i \atop kl} \right\}\left\{ {s \atop kl} \right\} = (b_i(k,l)\left\{ {i \atop kl} \right\})(b_s(k,l)\left\{ {s \atop kl} \right\}) = \phi_{kl}^2$ holds, Eq. (91) becomes

$$\left((a(k,l)-1)\partial_l - \sum_{m\neq l}\partial_m\right)\phi_{kl} + \phi_{kl}^2 = \lambda(k,l)\phi_{kl}. \tag{92}$$

With the normalised orthogonal system of the q hermetric coordinates $\vec{e}_i\vec{e}_k = \delta_{ik}$ and $\vec{a}_{kl} = \vec{e}_l(a(k,l)-1) - \sum_{m\neq l}\vec{e}_m$ Eq. (92) can be written as

$$\vec{a}_{kl}\vec{\nabla}_q\phi_{kl} = \lambda(k,l)\phi_{kl} - \phi_{kl}^2 = \frac{1}{4}\lambda^2(k,l)\left(1 - \left(\frac{2\phi_{kl}}{\lambda(k,l)} - 1\right)^2\right) \tag{93}$$

with $\vec{\nabla}_q$ being the gradient in the q coordinates and having reformed the r.h.s. so that a variable $u = \pm(\frac{2\phi_{kl}}{\lambda(k,l)} - 1)$ can be introduced. If we furthermore apply an inverse of the $(\vec{a}_{kl})^{-1} = \vec{a}_{kl}^{-1}$ and multiply with the differential $\mathrm{d}\vec{x} = \sum_{i=1}^{q}\vec{e}_i\mathrm{d}x_i$ we get

$$\mathrm{d}\vec{x}\,\vec{\nabla}_q u = \pm\frac{1}{2q}\lambda(k,l)\mathrm{d}\vec{x}\,\vec{a}_{kl}^{-1}(1-u^2) = \pm\frac{1}{2q}\lambda(k,l)(1-u^2)\sum_{i=1}^{q}\mathrm{d}x_i\left(a_{kl}^{-1}\right)_i = \pm\frac{1}{2}\lambda(k,l)(1-u^2)\mathrm{d}N_{kl} \tag{94}$$

with $dN_{kl} = \frac{1}{q}\sum_{i=1}^{q} dx_i\left(a_{kl}^{-1}\right)_i$.[44] Having a total differential on the l.h.s. we can transform to

$$\frac{du}{1-u^2} = \pm\frac{1}{2}\lambda(k,l)dN_{kl},$$

or with the primitive on the l.h.s.

$$\frac{1}{2}d\,\ln\left(\frac{1+u}{1-u}\right) = \pm\frac{1}{2}\lambda(k,l)dN_{kl}. \tag{95}$$

Integration gives

$$\ln\left(\frac{1+u}{1-u}\right) - \ln C_{kl} = \pm\lambda(k,l)N_{kl} \quad \text{or}$$
$$\frac{1+u}{1-u} = C_{kl}\,e^{\pm\lambda(k,l)N_{kl}} \tag{96}$$

with the integration constant C_{kl}. With the definition of $\psi_{kl} = \phi_{kl}/\lambda(k,l)$ and u as above we can write for the negative branch of u

$$u = -(2\psi_{kl} - 1), \quad \text{which gives}$$
$$\frac{1+u}{1-u} = \frac{1-2\psi_{kl}+1}{1+2\psi_{kl}-1} = \frac{1-\psi_{kl}}{\psi_{kl}} = \frac{1}{\psi_{kl}} - 1 = C_{kl}\,e^{-\lambda(k,l)N_{kl}}. \tag{97}$$

This finally gives the result

$$\psi_{kl} = (1 + C_{kl}\,e^{-\lambda(k,l)N_{kl}})^{-1} \tag{98}$$

which is already close to Eq. (32), but we still have to evaluate the N_{kl} and consider the positive branch of u. After an analogous calculation as (97) this branch gives the same result as (98), but only with an inverse constant $1/C_{kl}$ in front of the e-function. With a value of $C_{kl} = -1$, found in Appendix G, this does not make a difference to (98). For the N_{kl} holds $N_{kl} = \frac{1}{q}\vec{x}\,\vec{a}_{kl}^{-1} = x\frac{1}{q}\vec{e}_x\,\vec{a}_{kl}^{-1}$ with x being the absolute value of the q-dimensional "position" vector $\vec{x}$ as in Subsection 3.4.4 ((32) and following). Note that the unit vector $\vec{e}_x = (\vec{r} + i\vec{\xi})/\sqrt{r^2 - \xi^2}$ is in general a complex-valued quantity. With the aggregations $\vec{\lambda}_{kl} = \frac{1}{q}\lambda(k,l)\,\vec{a}_{kl}^{-1}$, respectively, $\epsilon_{kl} = \frac{1}{q}\vec{e}_x\,\vec{a}_{kl}^{-1} = \frac{1}{q}\vec{e}_x\,(\vec{e}_l(a(k,l)-1)^{-1} - \sum_{m\neq l}\vec{e}_m)$, which contains the angles of the 6-dimensional $\vec{x}$ in the unit vector $\vec{e}_x$, we get

$$N_{kl} = \epsilon_{kl}x, \quad \text{and so}$$
$$\psi_{kl} = \left(1 + C_{kl}\,e^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1} = (1 + C_{kl}\,e^{-\lambda(k,l)\epsilon_{kl}x})^{-1} \tag{99}$$

which provides the result stated in Eq. (32) when denoting $\lambda(k,l)\epsilon_{kl} = \lambda_{kl}$. We shortly want to note two important properties of the solution:

The extrema of (99) can immediately be derived from Eq. (92) as if the partial derivations become $\partial\phi_{kl} = 0$, then $\phi_{kl}^2 = \lambda(k,l)\phi_{kl}$ holds which gives the solutions $\phi_{kl} = 0$ or $\phi_{kl} = \lambda(k,l)$, or expressed with ψ_{kl}, $\psi_{kl} = 0$ or $\psi_{kl} = 1$.

The second point is the question whether the general covariance of Eq. (30) and its solution can be shown (which was not explicitly considered by Heim). For this we realise again that the solution of Eq. (92) solves (30) and that we get the expression $A(k,l) = \lambda(k,l)^2\psi_{kl}$ on both sides of (92) when inserting the solution (99), respectively (98). This expression depends on the coordinates x_i only in scalar form which means that it transforms as $\frac{\partial A}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'}\frac{\partial A}{\partial x_j}$ due to the derivation chain rule, i.e. transforms covariant.

F.2 Calculation of the metric tensor

Next, the metric tensor in the R_6 shall be calculated from the result ψ_{kl}. To achieve this, Heim derives from $\psi_{kl} = \frac{b_i(k,l)}{\lambda(k,l)}\left\{ {i \atop kl} \right\}$ (32) that

$$\psi_{kl} = \left(\sum_{s=1}^{q}\vec{e}_s\,\frac{b_s(k,l)}{\lambda(k,l)}\right)\vec{e}_i\left\{ {i \atop kl} \right\} = \frac{\vec{b}(k,l)}{\lambda(k,l)}\vec{e}_i\left\{ {i \atop kl} \right\} \tag{100}$$

and thus, by summing over the indices k,l, he defines a component

$$\psi_i = \sum_{k,l=1}^{q}\frac{\lambda(k,l)}{b_i(k,l)}\psi_{kl} = \sum_{k,l=1}^{q}\left\{ {i \atop kl} \right\}. \tag{101}$$

With the abbreviations $\sum_{k=1}^{q} g_{ik} = g_i$ and $\sum_{i=1}^{q} g_i = g$[45] and with the well-known relation between the Christoffel symbols and g_{ik} (see e.g. [52]) we get[46]

44 The factor $1/q$ results from the fact that in the step from equation (93) to (94) we fragment the scalar of (93) into q parts, which requires a division of the r.h.s. in (94) by q.

45 g may not be mixed up with the determinant of g_{ik}. Different from Heim, who uses the symbol γ_{ik} for the metric tensor (structure selector or fundamental selector in his terminology) to express that it has a different mathematical character in his first and second area of validity with finite metrons, we keep the notation g_{ik} of GR which makes sense in the third and fourth area of validity which we consider only.

46 As always, if not defined explicitly differently, it is summed over identical indices.

$$\psi_i = g^{is}\sum_{k,l=1}^{q}\left\{ {}_{skl} \right\}$$
$$= \frac{1}{2}g^{is}\sum_{k,l=1}^{q}(\partial_k g_{sl} + \partial_l g_{ks} - \partial_s g_{kl})$$
$$= \frac{1}{2}g^{is}\left(\sum_{k=1}^{q}\partial_k g_s + \sum_{l=1}^{q}\partial_l g_s - \partial_s g\right)$$
$$= \frac{1}{2}g^{is}\left(2\left(\sum_{k=1}^{q}\partial_k\right)g_s - \partial_s g\right). \tag{102}$$

Summation over i gives (with $\underline{\partial} = \sum_k \partial_k$)

$$\sum_{i=1}^{q} 2\psi_i g_i = \sum_{i,s=1}^{q} g^{is}(2\underline{\partial} g_s - \partial_s g)\sum_{l=1}^{q} g_{il}$$
$$= \sum_{l,s=1}^{q}\delta_{sl}(2\underline{\partial} g_s - \partial_s g)$$
$$= \sum_{s,k=1}^{q}(2\partial_k g_s - \partial_s g_k)$$
$$= \sum_{s,k=1}^{q}\partial_k g_s$$
$$= \underline{\partial} g = \sum_{i=1}^{q}\underline{\partial} g_i, \qquad \text{or} \qquad \sum_{i=1}^{q}(2\psi_i g_i - \underline{\partial} g_i) = 0. \tag{103}$$

This leads to

$$2\psi_i = \sum_{k=1}^{q}\frac{\partial_k g_i}{g_i} = \sum_{k=1}^{q}\vec{e}_k\,\frac{\vec{\nabla} g_i}{g_i}$$
$$= \vec{Z}\vec{\nabla}\ln g_i \quad \Rightarrow \quad \vec{\nabla}\ln g_i = \frac{2}{q}\vec{Z}^{-1}\psi_i \tag{104}$$

with $\vec{Z} = \vec{Z}^{-1} = \sum_k \vec{e}_k$. Here we have made a similar step as in (94), i.e. fragmented into q single equations through the gradient, thus the factor $1/q$ appears on the r.h.s. Multiplying with $\mathrm{d}\vec{x}$ and defining $\mathrm{d}\mu = \mathrm{d}\vec{x}\vec{Z}/q = \sum_k \mathrm{d}x_k/q$ gives

$$\mathrm{d}\vec{x}\,\vec{\nabla}\ln g_i = 2\mathrm{d}\mu\,\psi_i, \quad d\ln g_i = 2\mathrm{d}\mu\,\psi_i,$$
and after integration $\quad g_i = A_i\exp\left(2\int \mathrm{d}\mu\,\psi_i\right),$

the A_i being constants. The exponent is

$$2\int \mathrm{d}\mu\,\psi_i = 2\int \mathrm{d}\mu\sum_{k,l=1}^{q}\left\{ {i \atop kl} \right\} = 2\sum_{k,l=1}^{q}\frac{\lambda(k,l)}{b_i(k,l)}\int \mathrm{d}\mu\,\psi_{kl}. \tag{105}$$

The integral in the exponent can be further evaluated as follows, at the same time using the abbreviation $\vec{\lambda}_{kl} = \frac{1}{q}\lambda(k,l)\vec{a}_{kl}^{-1}$:

$$\int \mathrm{d}\mu\,\psi_{kl} = \int\sum_{s=1}^{q}\mathrm{d}x_s\,\frac{1}{q}\psi_{kl}$$
$$= \frac{1}{q}\sum_{s=1}^{q}\int\left(1 + C_{kl}\,\mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1}\mathrm{d}x_s,$$
$$\int\left(1 + C_{kl}\,\mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1}\mathrm{d}x_s = \int\left(1 + C_{kl}\,\mathrm{e}^{-\sum_p \lambda_p x_p}\right)^{-1}\mathrm{d}x_s$$
$$= \int\left(1 + C_{kl}\,\mathrm{e}^{-\sum_{p\neq s}\lambda_p x_p - \lambda_s x_s}\right)^{-1}\mathrm{d}x_s$$
$$= \frac{1}{\lambda_s}\int(1 + \mathrm{e}^{-\eta})^{-1}\mathrm{d}\eta$$
$$= \frac{1}{\lambda_s}\int\frac{\mathrm{e}^{\eta}}{\mathrm{e}^{\eta} + 1}\mathrm{d}\eta$$
$$= \frac{1}{\lambda_s}\int\frac{d(\mathrm{e}^{\eta} + 1)}{\mathrm{e}^{\eta} + 1}$$
$$= \frac{1}{\lambda_s}\ln(\mathrm{e}^{\eta} + 1) + c$$
$$= \frac{1}{\lambda_s}\ln\left(B_{kl}\left(\frac{\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}}}{C_{kl}} + 1\right)\right) \tag{106}$$

Here we used the substitution $\eta = -\ln C_{kl} + \sum_{p\neq s}\lambda_p x_p + \lambda_s x_s$, introduced the integration constant B_{kl} and partially suppressed the indices k, l. The integral $\int \mathrm{d}\mu\,\psi_i$ now gives

$$\int \mathrm{d}\mu\,\psi_i = \sum_{k,l=1}^{q}\frac{\lambda(k,l)}{q b_i(k,l)}\sum_{s=1}^{q}\frac{1}{\lambda_s}\ln\left(B_{kl}\left(\frac{\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}}}{C_{kl}} + 1\right)\right) \tag{107}$$

where the sum over the $1/\lambda_s$ and the argument of the ln can be further evaluated as

$$\vec{\lambda}_{kl} = \frac{\lambda(k,l)}{q}\left(\frac{1}{a(k,l) - 1}, -1, \ldots, -1\right),$$
$$\Rightarrow \sum_{s=1}^{q}\frac{1}{\lambda_s} = \frac{q}{\lambda(k,l)}(a(k,l) - 1 - (q - 1))$$
$$= \frac{q}{\lambda(k,l)}(a(k,l) - q),$$
$$C_{kl}\,\mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}} = \frac{1}{\psi_{kl}} - 1, \quad \Rightarrow \quad \frac{\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}}}{C_{kl}} + 1 = (1 - \psi_{kl})^{-1}, \tag{108}$$

leading to

$$\int \mathrm{d}\mu\,\psi_i = \sum_{k,l=1}^{q}\frac{a(k,l) - q}{b_i(k,l)}\ln\left(\frac{B_{kl}}{1 - \psi_{kl}}\right). \tag{109}$$

With $c_i(k,l) = (a(k,l) - q)/b_i(k,l)$ the contracted metric tensor becomes[47]

$$\sum_{k=1}^{q} g_{ik} = g_i = A_i\, \mathrm{e}^{2\sum_{k,l=1}^{q} c_i(k,l)\,\ln\left(\frac{B_{kl}}{1-\psi_{kl}}\right)}$$

$$= A_i \prod_{k,l=1}^{q} \left(\frac{1-\psi_{kl}}{B_{kl}}\right)^{-2c_i(k,l)}, \qquad (110)$$

or with the explicit form of the ψ_{kl}

$$\sum_{k=1}^{q} g_{ik} = A_i \prod_{k,l=1}^{q} \left(B_{kl}\left(\frac{\mathrm{e}^{\frac{1}{q}\lambda(k,l)\vec{a}_{kl}^{-1}\vec{x}}}{C_{kl}} + 1\right)\right)^{2c_i(k,l)}. \qquad (111)$$

F.3 Relation for a condenser anti-hermetric in the contravariant index

Finally we want to give a relation which Heim derived for a condenser (Christoffel symbol) which is anti-hermetric in the contravariant index. According to the anti-hermetry conditions as given at the beginning of subsection 3.4.4, all condenser components with covariant anti-hermetric indices vanish. This does not hold for a contravariant anti-hermetric index. But due to the cited relations the coefficients $b_{i=\tilde{r}}(k,l)$ are not defined in this case. Nonetheless, a solution for the non-vanishing contravariant anti-hermetric condenser components can be obtained in the following way:

If Eq. (30) is summed over index m, giving (91), and the contravariant index $i = \tilde{r}$ is anti-hermetric, then due to $b_s(k,l)\left\{ {s \atop kl} \right\} = \phi_{kl} = \lambda(k,l)\psi_{kl}$ (with sum over the hermetric index s) it follows

$$\left((a(k,l)-1)\,\partial_l - \sum_{m\neq l}\partial_m\right)\left\{ {\tilde{r} \atop kl} \right\} = \lambda(k,l)(1-\psi_{kl})\left\{ {\tilde{r} \atop kl} \right\} \qquad (112)$$

with ψ_{kl} being the solution of the hermetric problem according to (32). We use again the vector $\vec{a}_{kl}$ and the transformation to q equations with the gradient

$$\vec{\nabla}\ln\left\{ {\tilde{r} \atop kl} \right\} = \frac{1}{q}\lambda(k,l)\vec{a}_{kl}^{-1}(1-\psi_{kl}) \quad \text{or}$$
$$d\ln\left\{ {\tilde{r} \atop kl} \right\} = \frac{1}{q}\lambda(k,l)\vec{a}_{kl}^{-1}(1-\psi_{kl})\mathrm{d}\vec{x}, \quad \text{thus} \qquad (113)$$
$$\left\{ {\tilde{r} \atop kl} \right\} = B_{kl}^{\tilde{r}}\,\exp\frac{1}{q}\lambda(k,l)\vec{a}_{kl}^{-1}\left(\vec{x} - \int \psi_{kl}\mathrm{d}\vec{x}\right).$$

47 These results were not explicitly specified in [43].

Appendix G: Analysis of Hermetry forms *a* and *b*

Here we summarise the main content of chapter IV.3, pages 212–225, in [43]:

In case of the form a, which we want to consider first, the general line element $x^2 = \sum_{i=1}^{q} x_i^2$ becomes $x_5^2 + x_6^2 = -\zeta^2$ as x_5 and x_6 are imaginary. According to the analysis in Subsection 3.4.4 there must be discrete point spectra $\lambda\zeta$ for a condensation with a lower limit $\varepsilon > 0$ for ζ, as otherwise there would be a continuous spectrum, contrary to the meaning of condensation. As only x_5 and x_6 are involved, there are only the condenser components $\left\{ {i \atop 55} \right\}, \left\{ {i \atop 56} \right\}$ and $\left\{ {i \atop 66} \right\}$ which are, as explained in Subsection 3.4.4 and Appendix F, interconnected through proportionalities so that we can define $\underline{\phi}_i = \left\{ {i \atop 55} \right\}, \underline{\phi}_i/a = \left\{ {i \atop 56} \right\}$ and $\underline{\phi}_i b/a = \left\{ {i \atop 66} \right\}$. If we further set $\lambda_1 = \lambda_6(5,5)$ and $\lambda_2 = \lambda_5(6,6)$, then the relations $a\lambda_1\underline{\phi}_i = (\partial_5 - a\partial_6)\underline{\phi}_i - \frac{\alpha}{a}\underline{\phi}_i\underline{\phi}_6$ and $b\lambda_2\underline{\phi}_i = (\partial_6 - b\partial_5)\underline{\phi}_i - \frac{\alpha}{a}\underline{\phi}_i\underline{\phi}_5$ are obtained with the factor $\alpha = ab - 1$. Using the abbreviations $\underline{\lambda} = a\lambda_1 + b\lambda_2$ and $\underline{\phi} = \underline{\phi}_5 + \underline{\phi}_6$, the in total 4 relations ($i = 5, 6$) can be added, providing $\underline{\lambda}\underline{\phi} = -((b-1)\partial_5 + (a-1)\partial_6)\underline{\phi} - \frac{\alpha}{a}\underline{\phi}^2$. Now, it can be assumed that $\underline{\phi}$ only depends on the two coordinates in the form $\underline{\phi}(x_5, x_6) = \underline{\phi}(\xi)$ with the new variable ξ, defined via the substitutions $x_5 = x(b-1)$, $x_6 = y(a-1)$ and $\frac{\mathrm{d}\underline{\phi}}{\mathrm{d}x} + \frac{\mathrm{d}\underline{\phi}}{\mathrm{d}y} = \frac{\mathrm{d}\underline{\phi}}{\mathrm{d}\xi}$ and thus not to be mixed up with $\vec{\xi}$, as defined in the other chapters of this paper, and with $\phi = \frac{\alpha}{a}\underline{\phi}$ yielding the equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \phi^2 = -\underline{\lambda}\phi, \qquad (114)$$

which formally holds for $\lambda = -\underline{\lambda}$, too. The equation can easily be solved by separation of variables, leading to the result $\phi = \lambda(1 - \frac{1}{C}\mathrm{e}^{-\lambda\xi})^{-1}$ with $C = 1 \to C_{kl} = -1$ in Eq. (32), fulfilling the convergence requirements.[48] As ξ is imaginary, the eigenvalues which provide a maximum of $\phi(=\lambda)$ correspond to those of the general problem described in subsection 3.4.4 ([43]). But, as the hermetry form a describes terms in x_5 and x_6 outside the R_4, a physical interpretation seems to be impossible. It is only

48 Heim does not explicitly state which convergence requirements should be fulfilled. But, as the above choice for C, C_{kl} is the only way to achieve a course of ϕ, respectively ψ_{kl}, which converges against ∞ for ξ or $\vec{x} \to 0$, his way of thought is obvious. It seems clear that the strength of condensation should increase to infinity for zero distances in a “curved” space. Note that it always remains finite, since there is a smallest distance, defined by the metron.

conceivable that the structure a has an impact on the R_4 if ξ depends on the coordinates of the R_4 in any form. This is the case if the structure a fulfils the condition of a null geodesic, because then the R_4-sector of the fundamental metric tensor is pseudo-Euclidian (due to its anti-hermetry) and the metric quantities of a, for $\mathrm{d}s^2 = 0$, depend on the R_4-coordinates. So, even if a is not directly physically explainable, its effect on the (anti-hermetric) R_4 can be analysed. Heim therefore undertakes an analysis of the condensers (Christoffel symbols) and the metric tensor components $g_{ik}, i, k = 5, 6$, using the algebraic properties $g_{ik} = g_{ik}^{\times}, div_6\, g_{ik} = 0$,[49] and derives the equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - \frac{1}{4}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)^2 + \frac{\lambda}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0 \tag{115}$$

with $\psi = \ln g_{55} + \ln g_{66}$, giving the solution $g_{55}\, g_{66} = B(1 + A\mathrm{e}^{-\frac{\lambda}{2}\xi})^{-4}$, A and B being integration constants. Carrying out the step to a null geodesic, this means $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}\xi^2 = 0$ with $\mathrm{d}r^2 = \sum_{k=1}^{4}\mathrm{d}x_k^2$ or $\mathrm{d}\xi = \mathrm{i}\mathrm{d}r$. (115) then becomes an equation in the R_4 spacetime

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} - \frac{1}{4}\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 + \frac{\mathrm{i}\lambda}{2}\frac{\mathrm{d}\psi}{\mathrm{d}r} = 0. \tag{116}$$

For the further consideration the quadratic term $(\mathrm{d}\psi/\mathrm{d}r)^2$ can be neglected (as it describes only interactions between similar null geodesic processes in the R_4). The remaining equation can be solved by the approximation $\mathrm{d}^2/\mathrm{d}r^2 \to \sum_{k=1}^{4}\partial^2/\partial x_k^2$ and by the separation $\psi = \vartheta(t)w(x_k)$, $1 \le k \le 3$. After a somewhat longer calculation (see [43], pages 219–222) the following result is obtained with $R = \sum_{k=1}^{3}x_k^2$ and $x_4 = Ct$

$$w(R) \sim (1 + A\,\mathrm{e}^{2\beta R})\,\mathrm{e}^{-\frac{i}{4}\lambda_R R - \beta R} \quad \text{with} \quad \beta = \sqrt{a - \frac{\lambda_R^2}{16}},$$

a being the separation constant,

$$\vartheta(t) \sim \frac{B}{\sqrt{F}}(1 + F)\,\mathrm{e}^{-i\lambda Ct/4} \quad \text{with} \quad F = A\,\mathrm{e}^{2\kappa t} \quad \text{and}$$

$$\kappa = iC\sqrt{\lambda^2/16 + a}, \tag{117}$$

the remaining constants being constants of integration. As ψ is a state function, it must converge, i.e. $\lim_{t,R\to\infty}\psi = \lim_{t\to\infty}\vartheta \lim_{R\to\infty} w < \infty$, which is fulfilled by the spatial function w if $\mathrm{Re}\beta = 0$ and $\mathrm{Im}\beta \ge 0$, because only then w becomes a complex periodic function. This provides a constraint for the separation constant $a < \lambda_R^2/16$. The run of ϑ, however, is determined by C which in the anti-hermetric R_4 can be $C = ic$ (with c the velocity of light) in the case of an Lorentz invariant Euclidian spacetime R_{-4} or can be a real value $C = \omega$. The case $C = ic$ gives $\lim_{t\to\infty}\vartheta = \infty$ as can be seen from (117), which means that the R_{-4} cannot be the anti-hermetric spacetime of the a form and the null geodesic process of ψ cannot appear in the R_4 as an electromagnetic radiation quantum (photon). In the real case $C = \omega$ (R_{+4}) ϑ instead becomes a complex oscillation law so that the convergence $\lim_{t\to\infty}\vartheta < \infty$ is fulfilled. So, the anti-hermetric realm can only be a R_{+4} in which only gravitational actions can be present. If the x_5, x_6 appear in the connection $x_5^2 + x_6^2 = -\zeta^2$, and the world lines of the a hermetry form are null geodesics, ψ obviously causes a gravitational disturbance. As the λ are discrete spectra in the a hermetry, too, these advancing gravitational fields must have the character of discrete quantum levels as well, which therefore are identified as gravitons by Heim.

In the Hermetry form b, the coordinates x_4, x_5 and x_6 are hermetric, i.e. all three imaginary coordinates, and this means that an analogous formalism as for the form a exists. So, the respective state function becomes

$$\psi = \ln(g_{44}\, g_{55}\, g_{66}) = \ln\left(B\left(1 + A\,\mathrm{e}^{-\frac{\lambda}{2}\xi}\right)^{-4}\right)$$
$$= \ln B - 4\ln\left(1 + A\,\mathrm{e}^{-\frac{\lambda}{2}\xi}\right). \tag{118}$$

We now calculate[50] the first and second derivative of ψ, using $\xi = \sqrt{x_4^2 + x_5^2 + x_6^2}$ and $\frac{\mathrm{d}\xi}{\mathrm{d}x_4} = \frac{x_4}{\xi} \to 1$, $\frac{\mathrm{d}^2\xi}{\mathrm{d}x_4^2} = \frac{1}{\xi}\left(1 - \frac{x_4^2}{\xi^2}\right) \to 0$ for $x_4 \to \xi$ (i.e. $x_5 = x_6 \to 0$) and the abbreviations $X = A\mathrm{e}^{-\frac{\lambda}{2}\xi}$ and $f = 1 + X$:

$$\begin{aligned}\frac{\partial\psi}{\partial x_4} &= 2A\lambda\mathrm{e}^{-\frac{\lambda}{2}\xi}f^{-1}\frac{\mathrm{d}\xi}{\mathrm{d}x_4} \;\to\; 2\lambda X f^{-1}\\ \frac{\partial^2\psi}{\partial x_4^2} &= -A\lambda^2\mathrm{e}^{-\frac{\lambda}{2}\xi}f^{-2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x_4}\right)^2 + 2A\lambda\mathrm{e}^{-\frac{\lambda}{2}\xi}f^{-1}\frac{\mathrm{d}^2\xi}{\mathrm{d}x_4^2}\\ &\to -\lambda^2 X f^{-2}\end{aligned} \tag{119}$$

If we assume $X \ll 1$ and therefore expand $\psi \approx -4A\,\mathrm{e}^{-\frac{\lambda}{2}\xi} = -4X$ linearly, we get $f \approx 1$ and the following relation between ψ and its second derivative

$$\frac{\partial^2\psi}{\partial x_4^2} = -\lambda^2 X = \frac{\lambda^2}{4}\psi. \tag{120}$$

49 giving amongst others $\partial g_{55}/\partial x^5 = \partial g_{66}/\partial x^6 = 0$

50 The subsequent calculations were not specified by Heim, only the final result.

This relation can now be transferred to the situation in the R_4 where, because of the anti-hermetric R_3 coordinates, geodesic null lines exist $ds^2 = dr^2 + dx_4^2 = dr^2 - c^2t^2 = 0$ on the light cone. The R_3 coordinates then occur together with the time coordinate as well-known Lorentz invariant term $\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. This leads to the result

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)P = \frac{\lambda^2}{4}P \tag{121}$$

with $P = \psi(t)W(x_1, x_2, x_3)$. If we identify the r.h.s. as a current term,[51] this differential equation describes the dispersion law of photons in the frame of Quantum Electrodynamics (QED) in which, according to the quantum dualism, the particle character retreats in favour of the wave character for $\lambda \approx 0$. Then $\Delta P - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}P = 0$ results approximately. This is the equation of a transversal wave field in the R_3 which propagates with the velocity of light, identical with the empirical law of electromagnetic waves (principle d1). Therefore, it can be concluded that the imaginary time condensations of the Hermetry form b describe photons, the imponderable quanta of the electromagnetic field ([43]).

Appendix H: Properties and results of the poly-metric geometry

H.1 Properties of g_{ik} and $\begin{bmatrix} i \\ kl \end{bmatrix}$

In this subsection we present content of chapter VI.2, pages 86–94, of [44]:

The vanishing of a condenser with an anti-hermetric covariant index $\tilde{k}$, $\begin{bmatrix} i \\ \tilde{k}l \end{bmatrix}_{(\mu\nu)}^{(\kappa\lambda)} = 0$, means that also the condenser with a completely covariant signature vanishes

$$\begin{bmatrix} i\tilde{k}l \end{bmatrix}_{(pq)} = \frac{1}{2}(\partial_{\tilde{k}}g_{il}^{(pq)} + \partial_l g_{\tilde{k}i}^{(pq)} - \partial_i g_{\tilde{k}l}^{(pq)}) = 0, \quad \text{thus}$$
$$\partial_{\tilde{k}}g_{il}^{(pq)} = \partial_i g_{\tilde{k}l}^{(pq)} - \partial_l g_{\tilde{k}i}^{(pq)}, \tag{122}$$

leading to results for the hermitian and the anti-hermitian part of the partial metric tensor $g^{(pq)} = g_+^{(pq)} + g_-^{(pq)}$:

$$\begin{aligned}
\partial_{\tilde{k}}g_{+il}^{(pq)} &= \partial_{\tilde{k}}\frac{1}{2}(g_{il}^{(pq)} + g_{il}^{(pq)\times}) = \partial_{\tilde{k}}\frac{1}{2}(g_{il}^{(pq)} + g_{li}^{(pq)*}) \\
&= \frac{1}{2}(\partial_i g_{\tilde{k}l}^{(pq)} - \partial_l g_{\tilde{k}i}^{(pq)} + \partial_l g_{\tilde{k}i}^{(pq)*} - \partial_i g_{\tilde{k}l}^{(pq)*}) \\
&= \frac{1}{2}(\partial_i(g_{\tilde{k}l}^{(pq)} - g_{\tilde{k}l}^{(pq)*}) - \partial_l(g_{\tilde{k}i}^{(pq)} - g_{\tilde{k}i}^{(pq)*})) = 0, \\
\partial_{\tilde{k}}g_{-il}^{(pq)} &= \partial_{\tilde{k}}\frac{1}{2}(g_{il}^{(pq)} - g_{il}^{(pq)\times}) = \partial_{\tilde{k}}\frac{1}{2}(g_{il}^{(pq)} - g_{li}^{(pq)*}) \\
&= \frac{1}{2}(\partial_i g_{\tilde{k}l}^{(pq)} - \partial_l g_{\tilde{k}i}^{(pq)} - \partial_l g_{\tilde{k}i}^{(pq)*} + \partial_i g_{\tilde{k}l}^{(pq)*}) \\
&= \partial_i g_{\tilde{k}l}^{(pq)} - \partial_l g_{\tilde{k}i}^{(pq)},
\end{aligned} \tag{123}$$

as $g_{\tilde{k}l}^{(pq)} = g_{\tilde{k}l}^{(pq)*}$ is real. This means that $g_+^{(pq)}$ is constant concerning $\tilde{k}$ and that $g_-^{(pq)}$ can be calculated by integration of $g_{\tilde{k}l}^{(pq)}$ over $\tilde{k}$, giving a function $A_l^{(pq)} = \int g_{\tilde{k}l}^{(pq)} dx_{\tilde{k}}$ and thus $g_{-il}^{(pq)} = \partial_i A_l^{(pq)} - \partial_l A_i^{(pq)} = (rot_{(x)}\vec{A}^{(pq)})_{il} := P_{il}^{(pq)}$. So, $g_-^{(pq)}$ can be expressed through a vector which we identify as a spin field vector and already introduced in Subsection 3.5.1, $\vec{A}^{(pq)} = \vec{\phi}^{(pq)}$.[52]

Next we can derive a further property of $g_+^{(pq)}$ with an anti-hermetric coordinate $\tilde{k}$: From the defintion of the $A_l^{(pq)}$ folllows

$$\begin{aligned}
\partial_l \int g_{\tilde{k}m}^{(pq)} dx_{\tilde{k}} &= \partial_l A_m^{(pq)} \quad \Rightarrow \quad \partial_l g_{\tilde{k}m}^{(pq)} = \partial_{\tilde{k}}\partial_l A_m^{(pq)} \\
&= \partial_l \partial_{\tilde{k}} A_m^{(pq)} \quad \text{or} \quad g_{\tilde{k}m}^{(pq)} = \partial_{\tilde{k}} A_m^{(pq)}
\end{aligned}$$

and with $g_{\tilde{k}m}^{(pq)} = g_{+\tilde{k}m}^{(pq)} + g_{-\tilde{k}m}^{(pq)} = g_{+\tilde{k}m}^{(pq)} + P_{\tilde{k}m}^{(pq)}$

$$\Rightarrow \quad g_{+\tilde{k}m}^{(pq)} + P_{\tilde{k}m}^{(pq)} = \partial_{\tilde{k}}\phi_m^{(pq)}.$$

With $P_{\tilde{k}m}^{(pq)} = (rot_{(x)}\vec{\phi}^{(pq)})_{\tilde{k}m}$ we get

$$g_{+\tilde{k}m}^{(pq)} = \partial_{\tilde{k}}\phi_m^{(pq)} - (rot_{(x)}\vec{\phi}^{(pq)})_{\tilde{k}m} = \partial_m \phi_{\tilde{k}}^{(pq)}. \tag{124}$$

So, we can summarise the results so far as[53]

$$\begin{aligned}
g_{+\tilde{k}m}^{(pq)} &= \text{const}^{53} = \partial_m \phi_{\tilde{k}}^{(pq)}, \\
g_{-km}^{(pq)} &= P_{km}^{(pq)} = (\text{rot}_{(x)}\vec{\phi}^{(pq)})_{km}.
\end{aligned} \tag{125}$$

Now, with $g_{+\tilde{k}m}^{(pq)}$ being hermitian, we can exchange indices and get with (125) $\partial_m \phi_{\tilde{k}}^{(pq)} = g_{+\tilde{k}m}^{(pq)} = g_{+m\tilde{k}}^{(pq)} = \partial_{\tilde{k}}\phi_m^{(pq)}$, thus

$$\partial_{\tilde{k}}\phi_m^{(pq)} - \partial_m \phi_{\tilde{k}}^{(pq)} = (\text{rot}_{(x)}\vec{\phi}^{(pq)})_{\tilde{k}m} = P_{\tilde{k}m}^{(pq)} = g_{-\tilde{k}m}^{(pq)} = 0,$$
$$\Rightarrow \quad g_{+\tilde{k}m}^{(pq)} = \partial_{\tilde{k}}\phi_m^{(pq)} = \text{const}. \tag{126}$$

51 In doing so, we have to be aware that P is not only the state function of the EM field, but also of the matter. A distinction or split into two parts as given in the field equations of the QED – into the Dirac matter field ψ and the EM field A_μ – is not yet given and still has to be found for a one by one comparison.

52 Different from Heim we define the spin vector as equal to A, i.e. without factor τ, the metron.

53 with respect to an anti-hermetric coordinate $\tilde{k}$.

If we integrate the last relation, we find $\phi_m = f(V_r)\, x_{\tilde{k}}$, with a function f of V_r, the space of the hermetric coordinates. Doing the same with $\partial_m \phi_{\tilde{k}} =$ const, where the constant must be the same as in the previous relation, as both being equal to the hermitian $= g_{+\tilde{k}m}$, we get $\phi_{\tilde{k}} = f(V_r)\, x_m$ and, as a generalisation, the expression $\phi_l = \sum_{m=1}^{6} f_{lm}(V_r)\, x_m$. A derivation of this expression with respect to an anti-hermetric coordinate gives a coefficient $f_{l\tilde{k}}$, as expected, but a partial derivation to a hermetric coordinate x_k yields not only f_{lk}, but additionally a second term $\sum_{m=1}^{6} \partial_k (f_{lm})\, x_m$, depending also on anti-hermetric coordinates $x_{m=\tilde{k}}$. This contradiction to our starting point $\partial_{\tilde{k}} \phi_m = \partial_m \phi_{\tilde{k}} =$ const with respect to $\tilde{k}$ can only be solved if the functions f_{lm} are constants $f_{lm} = a_{lm}$. This leads to

$$g_{+\tilde{k}l} = \partial_{\tilde{k}} \phi_l = a_{l\tilde{k}} = a_{\tilde{k}l} = \text{const} \tag{127}$$

while in the hermetric space V_r always

$$g_{+kl} = g_{+kl}(V_r),$$

$$\text{but} \quad g_{-kl} = P_{kl} = (\text{rot}_{(x)} \vec{\phi})_{kl} = a_{kl} - a_{lk} = \text{const} \neq 0 \tag{128}$$

holds (we suppressed the indices (pq) of the partial structures in the last expressions), as in this case in general $a_{kl} \neq a_{lk}$. The obtained relations for the metric tensor in the poly-metric mean that its hermitian part $g_+^{(pq)}$ is constant in an anti-hermetric sector, but is a structure function in the V_r of the hermetric coordinates. Instead, the anti-hermitian part $g_-^{(pq)}$, identified as a spin field tensor P, completely vanishes in the anti-hermetric sector and exists as a constant quantity in the hermetric V_r.

With these results we have derived all relations presented in subsection 3.5.1 in (41) and can conclude from them that the condensers in the poly-metric are hermitian: Their purely covariant form is given as

$$\begin{aligned} 2\begin{bmatrix} skl \end{bmatrix}_{(\mu\nu)} &= \partial_k g_{sl} + \partial_l g_{ks} - \partial_s g_{kl} \\ &= \partial_k g_{+sl} + \partial_l g_{+ks} - \partial_s g_{+kl} \\ &= 2\begin{bmatrix} skl \end{bmatrix}_{(\mu\nu)+}, \end{aligned} \tag{129}$$

because always $g_{-kl} =$ const or $g_{-\tilde{k}l} = 0$ holds. But this also means

$$g^{is}_{(\kappa\lambda)} \begin{bmatrix} skl \end{bmatrix}_{(\mu\nu)} = \begin{bmatrix} i \\ kl \end{bmatrix}^{(\kappa\lambda)}_{(\mu\nu)+} \quad \text{or} \quad \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}} = \widehat{\begin{bmatrix} \kappa\lambda \\ \mu\nu \end{bmatrix}}_+, \tag{130}$$

as the covariant symmetry is not effected by an anti-hermitian part in the contravariant signature.

H.2 Solution of equation (43)

We subsequently refer to the content of chapter VI.3, pages 108–113, of [44], but note that we calculate different results than Heim at some important points. Especially the function $f(k,l)$ is not introduced by Heim, i.e. remains equal 1 in his calculation.

As for the $\underline{\lambda}_m(l,m)$ and $\underline{\lambda}_l(m,m)$[54] the same identities hold as in the case of the composition field (see Appendix F), we obtain similar relations for the F^i_{ml} as for the $\left\{ {i \atop ml} \right\}$ of the composition field, $F^i_{km} = \underline{a}_m(k,l) F^i_{kl}$ as well as $F^i_{sl} = a_{(1)} F^i_{kl}$ and $F^i_{sm} = a_{(2)} F^i_{kl}$. This leads to

$$\begin{aligned} &(\underline{a}_m(k,l)\partial_l - \partial_m) F^i_{kl} + (a_{(1)} \begin{bmatrix} s \\ km \end{bmatrix} - a_{(2)} \begin{bmatrix} s \\ kl \end{bmatrix}) F^i_{kl} \\ &\quad = \underline{\lambda}_m(k,l) F^i_{kl}. \end{aligned} \tag{131}$$

Fully analogous proportionalities hold for the components of the $[\ ]$ so that $a_{(1)} \begin{bmatrix} s \\ km \end{bmatrix} - a_{(2)} \begin{bmatrix} s \\ kl \end{bmatrix} = \underline{b}_{ms}(k,l) \begin{bmatrix} s \\ kl \end{bmatrix}$ can always be set, which after summing over the hermetric indices $1 \leq m \leq q$ and with the abbreviations $\underline{a}(k,l) = \sum_{m=1}^{q} \underline{a}_m(k,l)$, $\underline{b}_s(k,l) = \sum_{m=1}^{q} \underline{b}_{ms}(k,l)$ and $\underline{\lambda}(k,l) = \sum_{m=1}^{q} \underline{\lambda}_m(k,l)$ provides

$$\begin{aligned} &\left((\underline{a}(k,l) - 1)\partial_l - \sum_{m \neq l} \partial_m \right) F^i_{kl} + \underline{b}_s(k,l) \begin{bmatrix} s \\ kl \end{bmatrix} F^i_{kl} \\ &\quad = \underline{\lambda}(k,l) F^i_{kl}. \end{aligned} \tag{132}$$

Note that for simplicity of the notation we have again, as in reference (43), suppressed the indices of the partial structures in the previous expressions, but that in general the underlined terms depend on these indices. Therefore, this will hold for subsequently underlined quantities, too.

With a normalised orthogonal system of the q hermetric coordinates and $\underline{\vec{a}} = \vec{e}_l(\underline{a}(k,l) - 1) - \sum_{m \neq l} \vec{e}_m$ the previous equation can be written as

$$\vec{\nabla}_q F^i_{kl} = \frac{1}{q} \underline{\vec{a}}^{-1} (\underline{\lambda}(k,l) - \underline{b}_s(k,l) \begin{bmatrix} s \\ kl \end{bmatrix}) F^i_{kl}, \tag{133}$$

similar as done in Appendix F (94). Also similarly multiplying with the differential $d\vec{x} = \sum_{i=1}^{q} \vec{e}_i dx_i$ we get

54 We mark the λ and the quantities derived from them of Eq. (43) by an underline to distinguish them from the respective quantities of the composition field, and we subsequently move from the $\left\{ {s \atop kl} \right\}$ to the $\begin{bmatrix} s \\ kl \end{bmatrix}$ in the notation, as we have expressed the composition field in the poly-metric this way, see (40), because it has to be understood as a superposition of poly-metric structure terms.

$$\mathrm{d}\vec{x}\,\vec{\nabla}_q F^i_{kl} = \mathrm{d}F^i_{kl} = \frac{1}{q}\sum_{j=1}^{q} \mathrm{d}x_j \underline{a}_j^{-1}(\underline{\lambda}(k,l) - \underline{b}_s(k,l)\begin{bmatrix} s \\ kl \end{bmatrix})F^i_{kl} \tag{134}$$

which can be integrated to

$$\begin{aligned} F^i_{kl} &= A^i_{kl}\exp\left(\frac{1}{q}\int\sum_{j=1}^{q}\mathrm{d}x_j\underline{a}_j^{-1}\left(\underline{\lambda}(k,l) - \underline{b}_s(k,l)\begin{bmatrix} s \\ kl \end{bmatrix}\right)\right) \\ &= A^i_{kl}\exp\left(\vec{\underline{\lambda}}_{kl}\vec{x} - \int\sum_{j=1}^{q}\mathrm{d}x_j\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix}\right) \end{aligned} \tag{135}$$

with the integration constant A^i_{kl} and the abbreviations $\vec{\underline{\lambda}}_{kl} = \frac{1}{q}\underline{\lambda}(k,l)\vec{\underline{a}}^{-1}$ and $\underline{c}_{js} = \frac{1}{q}\underline{a}_j^{-1}\underline{b}_s(k,l)$. This result means that the problem of solving the fundamental equation of the poly-metric can be reduced to an integral over the condenser (state function) of the composition field. We therefore insert that result, given in Eq. (32), in the form $b_s(k,l)\left\{\begin{smallmatrix} s \\ kl \end{smallmatrix}\right\} = b_s(k,l)\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right] = \lambda(k,l)(1 - e^{-\frac{1}{q}\lambda(k,l)\vec{a}^{-1}\vec{x}})^{-1}$. To derive a needed expression for a single $\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]$ (instead of the sum over the index s), we again use the fragmentation into q parts on the r.h.s. (as done in references (94), (104) and (133)) and get $\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right] = \frac{\lambda(k,l)}{qb_s(k,l)}(1 - \mathrm{e}^{-\frac{1}{q}\lambda(k,l)\vec{a}^{-1}\vec{x}})^{-1}$. The integrand then becomes $\underline{c}_{js}\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right] = \frac{1}{q^2}\underline{a}_j^{-1}\vec{\underline{b}}(k,l)\vec{b}^{-1}(k,l)\lambda(k,l)(1 - \mathrm{e}^{-\frac{1}{q}\lambda(k,l)\vec{a}^{-1}\vec{x}})^{-1} = \underline{\omega}_j(k,l)\left(1 - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1}$, where we have expressed the sum over index s as the scalar product between $\vec{\underline{b}}$ and $\vec{b}^{-1}$. In the last step we have introduced the abbreviations $\vec{\lambda}_{kl} = \frac{1}{q}\lambda(k,l)\vec{a}^{-1}$ and $\underline{\omega}_j(k,l) = \frac{1}{q^2}\underline{a}_j^{-1}\vec{\underline{b}}(k,l)\vec{b}^{-1}(k,l)\lambda(k,l)$. The integral can be calculated as follows, using the same substitution as in reference (106)

$$\begin{aligned} \int\sum_{j=1}^{q}\mathrm{d}x_j\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix} &= \sum_{j=1}^{q}\underline{\omega}_j(k,l)\int\mathrm{d}x_j\left(1 - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1} \\ &= \ln\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1\right)\sum_{j=1}^{q}\frac{\underline{\omega}_j(k,l)}{\lambda^j_{kl}} + \text{const} \end{aligned}$$

with the factor
$$\sum_{j=1}^{q}\frac{\underline{\omega}_j(k,l)}{\lambda^j_{kl}} = \vec{\underline{b}}(k,l)\vec{b}^{-1}(k,l) \times\left(\frac{\underline{a}(k,l) - a(k,l)}{a(k,l) - 1} + q\right)^{-1} := \underline{\alpha}_{kl} \tag{136}$$

providing[55]

$$\int\sum_{j=1}^{q}\mathrm{d}x_j\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix} = \ln\left(B_{kl}\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1\right)^{\underline{\alpha}_{kl}}\right) \tag{137}$$

with B_{kl} being a constant. Note that the exponent $\underline{\alpha}_{kl}$ can become positive or negative, depending on the coefficients $\underline{a}(k,l)$ and $\vec{\underline{b}}(k,l)$, and that it becomes $= 1$ if these coefficients become equal to their analogous quantities of the composition field.[56] We now insert this result into equation (135), merge the constants $A^i_{kl}B^{-1}_{kl} = C^i_{kl}$ and obtain

$$F^i_{kl} = C^i_{kl}\,\mathrm{e}^{\vec{\underline{\lambda}}_{kl}\vec{x}}\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1\right)^{-\underline{\alpha}_{kl}}. \tag{138}$$

This result can be analysed by identifying the extrema with the condition $\partial F^i_{kl} = 0$ which is already defined via Eq. (134). It can only be fulfilled by $F^i_{kl} = 0$ or $\underline{\lambda}(k,l) = \underline{b}_s(k,l)\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]$. $F^i_{kl} = 0$ leads again to an eigenvalue problem if $\vec{x}$ contains imaginary parts, which is identical to that of the composition field. The other condition means, as $\underline{\lambda}(k,l) = \text{const}$ and $\underline{b}_s(k,l) = \text{const} \neq 0$, $\partial\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right] = 0$, thus $\underline{\lambda}(k,l) = \underline{b}_s(k,l)\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]_{\text{ext}}$, but also $\lambda(k,l) = b_s(k,l)\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]_{\text{ext}}$ which directly can be seen from the expression (32) for the composition field and its maxima ($\psi = 1$). We again have equations with sums over an index s which cannot simply be solved for a single $\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]_{\text{ext}}$. But both equations can be fulfilled if they are fragemented into q parts and the single terms equaled ($b_s(k,l)\left[\begin{smallmatrix} s \\ kl \end{smallmatrix}\right]_{\text{ext}} = \lambda(k,l)/q$ for a single s, analogical for the underlined terms). This leads to $\frac{\underline{\lambda}(k,l)}{\lambda(k,l)} = \frac{\underline{b}_{(s)}(k,l)}{b_{(s)}(k,l)}$ for each s, or after summation over s to $\underline{\lambda}(k,l) = \frac{\lambda(k,l)}{q}\sum_{s=1}^{q}\frac{\underline{b}_s(k,l)}{b_s(k,l)} = \lambda(k,l)\frac{\vec{\underline{b}}(k,l)\vec{b}^{-1}(k,l)}{q}$. Thus $\underline{\lambda}(k,l) = \lambda(k,l)$ holds if $\underline{b}_s(k,l) = b_s(k,l)$.

We now consider the poly-metric state function $\underline{\psi}_{kl}$ and transform it as

$$\begin{aligned} \underline{\psi}_{kl} &= \mathrm{e}^{\vec{\underline{\lambda}}_{kl}\vec{x}}\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1\right)^{-\underline{\alpha}_{kl}} \\ &= \left(\mathrm{e}^{-\frac{\vec{\underline{\lambda}}_{kl}}{\underline{\alpha}_{kl}}\vec{x}}\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1\right)\right)^{-\underline{\alpha}_{kl}} \\ &= \left(\mathrm{e}^{\left(\vec{\lambda}_{kl} - \frac{\vec{\underline{\lambda}}_{kl}}{\underline{\alpha}_{kl}}\right)\vec{x}} - \mathrm{e}^{-\frac{\vec{\underline{\lambda}}_{kl}}{\underline{\alpha}_{kl}}\vec{x}}\right)^{-\underline{\alpha}_{kl}}. \end{aligned} \tag{139}$$

55 The exact term of the exponent was found by inserting the result (138) into the differential equation (132) and obtaining an equation for $\underline{\alpha}_{kl}$, because the algebraic evaluation in (136) remains ambiguous in the sum $\sum_j a^j \underline{a}_j^{-1}$.

56 If $\vec{\underline{b}}_s(k,l) = \vec{b}_s(k,l)$ then $\vec{\underline{b}}(k,l)\vec{b}^{-1}(k,l) = \sum_{s=1}^{q}\frac{\underline{b}_s(k,l)}{b_s(k,l)} = \sum_{s=1}^{q} = q$.

The analysis of the exponent of the first term gives

$$\vec{\lambda}_{kl} - \frac{\vec{\underline{\lambda}}_{kl}}{\underline{\alpha}_{kl}} = \frac{\lambda(k,l)}{q}\left(\vec{a}^{-1} - \underline{\vec{a}}^{-1}\frac{1}{q}\left(\frac{\underline{a}(k,l)-a(k,l)}{a(k,l)-1}+q\right)\right)$$
$$\rightarrow \frac{\lambda(k,l)\vec{a}^{-1}}{q}\left(1-\frac{q}{q}\right) = 0 \qquad (140)$$

if $\underline{a}(k,l) = a(k,l)$. Note that the dependency on the $\underline{b}_s(k,l)b_s^{-1}(k,l)$ cancels out, so only the $\underline{a}(k,l)$ need to be $= a(k,l)$ to obtain $\vec{\underline{\lambda}}_{kl} = \underline{\alpha}_{kl}\vec{\lambda}_{kl}$. Important to see is that this equation does not hold in general, different as Heim specified it. If it holds, then the poly-metric state function can be written as

$$\underline{\psi}_{kl} = \left(1-\mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-\underline{\alpha}_{kl}} = \psi_{kl}^{\underline{\alpha}_{kl}}. \qquad (141)$$

As next we can derive another important relation by considering again the extrema of our solution. Applying the derivation $\partial F^i_{kl} = 0$ directly to (138) we find $\vec{\underline{\lambda}}_{kl} = \mathrm{e}^{\vec{\lambda}_{kl}\vec{x}}(\vec{\underline{\lambda}}_{kl} - \underline{\alpha}_{kl}\vec{\lambda}_{kl})$ after a few transformations. If we build the scalar product with the vectors $\vec{r}$ and $\vec{\xi}$ (of $\vec{x} = \vec{r} + i\vec{\xi}$) and divide these relations, we get

$$\frac{\vec{r}\,\vec{\underline{\lambda}}_{kl}}{\vec{r}\,\vec{\underline{\lambda}}_{kl} - \underline{\alpha}_{kl}\vec{r}\,\vec{\lambda}_{kl}} = \frac{\vec{\xi}\,\vec{\underline{\lambda}}_{kl}}{\vec{\xi}\,\vec{\underline{\lambda}}_{kl} - \underline{\alpha}_{kl}\vec{\xi}\,\vec{\lambda}_{kl}} \qquad (142)$$

as equation which (only) holds at the extrema $\vec{x}_{\mathrm{ext}}$, i.e. $\vec{r}_{\mathrm{ext}}$ and $\vec{\xi}_{\mathrm{ext}}$ of F^i_{kl}. This equation can definitely be fulfilled by $(\vec{r}\,\vec{\underline{\lambda}}_{kl})_{\mathrm{ext}} = d\cdot(\vec{\xi}\,\vec{\underline{\lambda}}_{kl})_{\mathrm{ext}}$ and $(\vec{r}\,\vec{\lambda}_{kl})_{\mathrm{ext}} = d\cdot(\vec{\xi}\,\vec{\lambda}_{kl})_{\mathrm{ext}}$ with a parameter or function d. The algebraic properties of d are considered in Subsection 4.2.[57]

We finally can derive an expression for $\underline{\alpha}_{kl}$ depending on the eigenvalues $\underline{\beta}_\pm$ of the poly-metric field and the $\beta_\pm$ of the composition field: We consider the arguments of the e-function in $\underline{\psi}_{kl}$ and ψ_{kl}

$$\frac{\vec{\underline{\lambda}}_{kl}\vec{x}}{\vec{\lambda}_{kl}\vec{x}} = \frac{\underline{\lambda}(k,l)\underline{\vec{a}}^{-1}\vec{x}}{\lambda(k,l)\vec{a}^{-1}\vec{x}}$$
$$= \frac{\underline{\lambda}(k,l)}{\lambda(k,l)}\frac{(\underline{a}(k,l)-1)^{-1}x_l - \sum_{m\neq l}x_m}{(a(k,l)-1)^{-1}x_l - \sum_{m\neq l}x_m}$$
$$:= \frac{\underline{\lambda}(k,l)}{\lambda(k,l)}\tilde{f}(k,l) \qquad (143)$$

with the function $\tilde{f}(k,l) \approx \frac{(\underline{a}(k,l)-1)^{-1}-q+1}{(a(k,l)-1)^{-1}-q+1}$ if the $x_{l/m}$ can be approximated in average as $x_{l/m} \approx x$, and $\tilde{f}(k,l) \rightarrow 1$ if $\underline{a}(k,l) = a(k,l)$. In this limit also $\underline{\lambda}(k,l)/\lambda(k,l) = \underline{\alpha}_{kl}$ holds (see above), but in general $\underline{\lambda}(k,l)/\lambda(k,l) = \underline{\alpha}_{kl}\,c(k,l)$ with $c(k,l) = \frac{\underline{a}(k,l)-a(k,l)}{q(a(k,l)-1)} + 1$ (compare reference (136)). This means[58]

$$\frac{\vec{\underline{\lambda}}_{kl}\vec{x}}{\vec{\lambda}_{kl}\vec{x}} = \underline{\alpha}_{kl}\,c(k,l)\tilde{f}(k,l) := \underline{\alpha}_{kl}f(k,l). \qquad (144)$$

On the other hand the ratio of the arguments for the extrema, i.e. eigenvalues (see subsection 3.5.2) is $(\vec{\underline{\lambda}}_{kl}\vec{x})_{\mathrm{ext}}/(\vec{\lambda}_{kl}\vec{x})_{\mathrm{ext}} = (\underline{a}_{\mathrm{ext}} + i\underline{\beta}_\pm)/(a_{\mathrm{ext}} + i\beta_\pm)$ with $\underline{a}$ and a being the real parts of the arguments. For the regime of the eigenvalues we can assume $y^2 = -x^2 = \xi^2 - r^2 > 0$ with $\vec{x} = \vec{r} + i\vec{\xi}$. If we now further apply $(\vec{\lambda}_{kl}\vec{r})_{\mathrm{ext}} = d\cdot(\vec{\lambda}_{kl}\vec{\xi})_{\mathrm{ext}}$ and the analogous relation for the $\vec{\underline{\lambda}}_{kl}$, as obtained above, we get

$$\frac{(\vec{\underline{\lambda}}_{kl}\vec{x})_{\mathrm{ext}}}{(\vec{\lambda}_{kl}\vec{x})_{\mathrm{ext}}} = \frac{(d+i)(\vec{\underline{\lambda}}_{kl}\vec{\xi})_{\mathrm{ext}}}{(d+i)(\vec{\lambda}_{kl}\vec{\xi})_{\mathrm{ext}}} = \frac{\underline{\beta}_\pm}{\beta_\pm}$$
$$\Rightarrow \quad \underline{\alpha}_{kl} = \frac{\underline{\beta}_\pm}{\beta_\pm}f^{-1}_{ext}(k,l) \quad \text{and} \quad \frac{\underline{\lambda}(k,l)}{\lambda(k,l)} = \frac{\underline{\beta}_\pm}{\beta_\pm}\tilde{f}^{-1}_{\mathrm{ext}}(k,l). \qquad (145)$$

These relations become relevant in Section 4.

57 Equation (142) and the resulting relations were not considered by Heim. These relations have a certain similarity to dispersion relations of waves which becomes more obvious in the limit of vanishing variables x_5 and x_6: A "phase" velocity of the extrema is $v_{ext} = r_{ext}/t_{ext} = d\,c\,\lambda_t/\lambda_r$, where λ_r, λ_t are the projections of $\vec{\lambda}_{kl}$ on $\vec{e}_r$, respectively, $\vec{e}_4$ at the extrema. In this context the definition of d can additionally be checked under kinematic aspects. We consider its defining relation $(\vec{\lambda}\vec{r})_{ext} = d\,(\vec{\lambda}\vec{\xi})_{ext} \Rightarrow \lambda\, r_{ext}\cos\theta^r_{ext} = d\,\lambda\,\xi_{ext}\cos\theta^\xi_{ext} \Rightarrow r_{ext} = \xi_{ext}\,d\cos\theta^\xi_{ext}/\cos\theta^r_{ext} := \xi_{ext}\,\tilde{d}_{ext}$, meaning also $r^2_{ext} = \tilde{d}^2_{ext}(c^2t^2 + \epsilon^2 + \eta^2)_{ext}$, if the coordinate definition of ξ is inserted. Applying the second time derivative and neglecting terms with accellerations and higher order terms in $\vec{\nabla}_6\tilde{d}$ (i.e. presuming $\tilde{d}$ being smooth and time dependent only via $\vec{r}, \vec{\xi}$) results in $v \approx 2r(\vec{\nabla}_6\tilde{d})\dot{\vec{x}}/\tilde{d} + \tilde{d}\,w$, with $v = \dot{r}$, the radial velocity in the R_3, and $w = \dot{\xi} = (c^2t + \epsilon\dot{\epsilon} + \eta\dot{\eta})/\xi$, the 'radial' imaginary part of the world velocity (compare subsection 3.4.5), all quantities to be taken at the extremal points of r and ξ. As the term with $\vec{\nabla}_6\tilde{d}$ can be positive or negative, though small for smooth little changing $\tilde{d}$, and w can vary around c (assuming that the velocities of the trans-coordinates are moderate), our assumption in subsection 4.2, $0.5 \leq d < 1$, is compatible to the constraints given by the approximate formula for v and the empirical fact that $v < c$ holds for massive particles. In nuclear matter there are momenta (of the nucleons) of about 250 MeV/c (see e.g. [75]), i.e. velocities of about $c/4$. In our picture of elementary particles, considered as condensations in the R_6 at a microscopic level smaller than nuclear dimensions, higher velocities closer to c seem to be likely so that values of d close to 1 appear to be possible as well.

58 As already stated in subsection 3.5.2, Heim instead derives a result without the modulating function $f(k,l)$, i.e. $f(k,l) = 1$, which corresponds to his (according to (140)) wrong finding that $\vec{\underline{\lambda}}_{kl} = \underline{\alpha}_{kl}\vec{\lambda}_{kl}$ would hold in general.

H.3 Expressions of the metric tensor and the correlation tensor in the poly-metric

Based on result (138), Heim obtains the following expression for the trace of the metric tensor

$$\sum_{k=1}^{q} g_{kk(\mu\nu)} = C_{(\mu\nu)} \prod_{l=1}^{q} \left(\mathrm{e}^{\vec{\lambda}_{ll}\vec{x}} - 1 \right)^{\underline{\alpha}_{ll}} = C_{(\mu\nu)} \left| \left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{x}} - 1 \right)^{\underline{\alpha}_{kl}} \delta_{kl} \right| \tag{146}$$

with integration constant $C_{(\mu\nu)}$ and the $\vec{\lambda}_{kl}, \underline{\alpha}_{kl}$ as being defined similarly in subsection H.2. $|\dots|$ denotes the determinant.

The correlation tensor results in the form of an invariant scalar sum as

$$\begin{aligned} Q_{(\mu\nu)}^{(\kappa\lambda)} &= \sum_{i,s=1}^{q} Q_{s(\mu\nu)}^{i(\kappa\lambda)} \\ &= A_{(\mu\nu)}^{(\kappa\lambda)} \sum_{l=1}^{q} C_{l(\mu\nu)}^{(\kappa\lambda)} \left(1 - \mathrm{e}^{-\vec{\lambda}_{ll}\vec{x}}\right)^{-\underline{\alpha}_{ll}} \\ &\times \left(\sum_{l=1}^{q} C_{l}^{(\mu\nu)} \left(1 - \mathrm{e}^{-\vec{\lambda}_{ll}^{(\mu\nu)}\vec{x}}\right)^{-1} \right)^{-1} \\ &\times \left| \delta_{kl} \left(\mathrm{e}^{\vec{\lambda}_{kl}^{(\kappa\lambda)}\vec{x}} - 1 \right)^{\alpha_{kl}^{(\kappa\lambda)}} \right| \\ &\times \left| \delta_{kl} \left(\mathrm{e}^{\vec{\lambda}_{kl}^{(\mu\nu)}\vec{x}} - 1 \right)^{\alpha_{kl}^{(\mu\nu)}} \right|^{-1} - q \end{aligned} \tag{147}$$

with $A_{(\mu\nu)}^{(\kappa\lambda)} = 2\, C_{(\kappa\lambda)} C_{(\mu\nu)}^{-1}$. Note that these expressions are only valid under the condition that (141) holds. The derivation of these results can be found on pages 113–121 in [44]. We do not present it here, since we do not use the expressions (146) and (147) in our further calculations.

H.4 Calculation of the condenser class [3]

We refer to chapter VII.1 of [44], pages 174–177:

To calculate the function $F_{(3)}$, defined by the general solution (44) and the definition $\mathrm{Im}\, F_{(3)} = \mathrm{Im}(\widehat{\begin{bmatrix} 33 \\ 33 \end{bmatrix}}_+ + \mathrm{Tr}(Q_{(33)}^{(33)} \times \widehat{\begin{bmatrix} 33 \\ 33 \end{bmatrix}}_+)) = 0$, we introduce the definitions $A_{kl(3)}^{i} = a + ib$ and $\sum_{j=1}^{3} \underline{c}_{js(3)} \int \mathrm{d}x_j \begin{bmatrix} s \\ kl \end{bmatrix} = f + iF$. With the always real $\underline{\vec{\lambda}}_{kl}$ and $\vec{x}$, $\mathrm{d}\vec{x}$ in the R_3 we get $0 = \mathrm{Im}((a+ib)\mathrm{e}^{\phi - \mathrm{i}F}) = \mathrm{e}^{\phi}\mathrm{Im}((a+ib)e^{-iF})$ with $\phi = \underline{\vec{\lambda}}_{kl}\vec{x} - f$ in the R_3. With $\mathrm{e}^{-\mathrm{i}F} = \cos F - i \sin F$ it follows $(a+ib)\mathrm{e}^{-\mathrm{i}F} = a\cos F + b \sin F + i(b\cos F - a \sin F)$, which with the $\mathrm{Im}() = 0$ condition yields $\tan F = b/a$ and $F_{(3)} = \mathrm{e}^{\phi}(a\cos F + b \sin F)$. Using the substitution with $\tan F$ we can write $\cos F = (1 + \tan^2 F)^{-1/2} = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin F = (1 + \cot^2 F)^{-1/2} = \frac{b}{\sqrt{a^2+b^2}}$, thus $a \cos F + b \sin F = \sqrt{a^2+b^2} := \alpha_{kl}^{i} = \text{const.}$ Thereby the real-valued solution becomes

$$F_{kl(3)}^{i} = \alpha_{kl}^{i}\mathrm{e}^{\phi} \quad \text{with} \quad \phi = \underline{\vec{\lambda}}_{kl}\vec{x} - \mathrm{Re}\left(\int \sum_{j=1}^{3} \mathrm{d}x_j \underline{c}_{js(3)} \begin{bmatrix} s \\ kl \end{bmatrix} \right) \tag{148}$$

with the coefficients $\underline{c}_{js(3)}$ being real quantities, as they are composed of real eigenvalues. Similar as already derived below reference (135), the integrand becomes $\underline{c}_{js(3)} \begin{bmatrix} s \\ kl \end{bmatrix} = \frac{1}{q^2} \underline{a}_{j(3)}^{-1} \underline{\vec{b}}_{(3)} \vec{b}^{-1} \lambda(k,l) \psi_{kl} = \omega_{j(3)}(k,l)\psi_{kl}$ with the abbreviation $\omega_{j(3)}(k,l)$ as coefficient and the state function ψ_{kl} of the composition field. ψ_{kl} is a function of the 6 coordinates of the R_6 as $\psi_{kl} = \left(1 - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{x}}\right)^{-1}$ with $\vec{x} = \vec{r} + \mathrm{i}\vec{\xi}$ and the $\vec{\lambda}_{kl}$, as defined for reference (136). The integral in (148) runs only over the three spatial coordinates $\vec{r}$ so that the imaginary coordinates $\vec{\xi}$ appear as constants in $\psi_{kl} = (1 - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{r}}(\cos \vec{\lambda}_{kl}\vec{\xi} - \mathrm{i} \sin \vec{\lambda}_{kl}\vec{\xi}))^{-1} = (\kappa + \mathrm{i}\eta)^{-1}$, having defined the quantities κ and η this way. According to this relation, also $\psi_{kl} = \alpha + i\beta$ must split, thus $\mathrm{Re}\, \psi_{kl} = \alpha$ and $f = \int \vec{\omega}_{(3)} \mathrm{d}\vec{r}\, \alpha$. As κ and η are known, β must be eliminated out of $\alpha + i\beta = (\kappa + i\eta)^{-1}$ in order to obtain α. Because of $1 = (\alpha + \mathrm{i}\beta)(\kappa + \mathrm{i}\eta) = \alpha\kappa - \beta\eta + \mathrm{i}(\alpha\eta + \kappa\beta)$, thus $\alpha\kappa - \beta\eta = 1$ and $\kappa\beta = -\alpha\eta$, we get $\alpha = \kappa(\kappa^2 + \eta^2)^{-1} = \left(1 - b\, \mathrm{e}^{-\vec{\lambda}_{kl}\vec{r}}\right)(1 - 2b\, \mathrm{e}^{-\vec{\lambda}_{kl}\vec{r}} + \mathrm{e}^{-2\vec{\lambda}_{kl}\vec{r}})^{-1}$ where we have used the abbreviation $b = \cos \vec{\lambda}_{kl}\vec{\xi}$. With $\vec{\omega}_{(3)}\mathrm{d}\vec{r} = \sum_{j=1}^{3} \omega_{j(3)} \mathrm{d}r_j$ and the substitution $u = \left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{r}} - b\right)/\sqrt{1-b^2}$, $\mathrm{d}r_j = \mathrm{d}u \frac{\sqrt{1-b^2}}{\lambda_{kl}^{j}} \mathrm{e}^{-\vec{\lambda}_{kl}\vec{r}}$ we can calculate the integral for f after some steps as

$$\begin{aligned} f &= \sum_{j=1}^{3} \frac{\omega_{j(3)}}{\lambda_{kl}^{j}} \int \frac{\mathrm{d}u\, u}{u^2 + 1} = \vec{\omega}_{(3)} \vec{\lambda}_{kl}^{-1} (\ln \sqrt{1 + u^2} + C) \\ &= \vec{\omega}_{(3)} \vec{\lambda}_{kl}^{-1} \left(\frac{1}{2} \ln (1 + \left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{r}} - b\right)^2 / (1 - b^2)) + C \right). \end{aligned} \tag{149}$$

The integration constant C can be determined by choosing $u(\vec{r} = 0)$ as lower limit of the integral above. Then C becomes

$$\begin{aligned} C &= -\frac{1}{2} \ln \left(1 + (1-b)^2/(1-b^2)\right) \\ &= -\frac{1}{2} \ln \left(1 + (1-b)/(1+b)\right) \\ &= \frac{1}{2} \ln \left(\frac{1}{2}(1+b)\right). \end{aligned} \tag{150}$$

This gives

$$f = \vec{\omega}_{(3)} \vec{\lambda}_{kl}^{-1} \frac{1}{2} \ln \left(\frac{1}{2}(1+b)(1 + \left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{r}} - b\right)^2 / (1 - b^2)) \right) \tag{151}$$

and for the factor

$$\vec{\omega}_{(3)}\vec{\lambda}_{kl}^{-1} = \frac{1}{q}\vec{a}\,\underline{\vec{a}}_{(3)}^{-1}\underline{\vec{b}}_{(3)}\vec{b}^{-1}$$
$$= \underline{\vec{b}}_{(3)}\vec{b}^{-1}\left(\frac{\underline{a}_{(3)}(k,l) - a(k,l)}{a(k,l)-1} + q\right)^{-1}$$
$$:= \underline{\alpha}_{(3)}. \quad (152)$$

in analogy to reference (136). The condenser function then becomes

$$F^i_{kl(3)} = \alpha^i_{kl}\,\mathrm{e}^{\underline{\vec{\lambda}}_{kl}\vec{r}}\left(\frac{1}{2}(1+b)\left(1 + \frac{\left(\mathrm{e}^{\,\vec{\lambda}_{kl}\vec{r}} - b\right)^2}{1-b^2}\right)\right)^{-\frac{\underline{\alpha}_{(3)}}{2}} \quad (153)$$

with $b = \cos\vec{\lambda}_{kl}\vec{\xi}$ (b not to be confused with $\vec{b} = \vec{b}(k,l)$).

H.5 Further classes of condensations

The first part of this subsection refers to chapter VII.1 of [44], pages 179–186.

Analogue to Subsection H.2 we can evaluate the expression of reference (54) by inserting the function of the composition field in the form $\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix} = \frac{1}{qq_v}\underline{a}_j^{-1}\underline{\vec{b}}(k,l)\vec{b}^{-1}(k,l)\lambda(k,l)(1-\ \mathrm{e}^{-\frac{1}{q}\lambda(k,l)\vec{a}^{-1}\vec{Q}})^{-1} := \underline{\omega}_j(k,l)$ $(1-\ \mathrm{e}^{-\,\vec{\lambda}_{kl}(\vec{P}+\vec{V})})^{-1}$. The integral in reference (54) then can be calculated as follows, using the same substitution as in reference (106) and taking into account $\vec{V} = 0$ at the lower limit of the integral over the V_j

$$\int\sum_{j=1}^{q_v}\mathrm{d}V_j\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix}$$
$$= \sum_{j=1}^{q_v}\underline{\omega}_j(k,l)\int\mathrm{d}V_j(1-\ \mathrm{e}^{-\,\vec{\lambda}_{kl}(\vec{P}+\vec{V})})^{-1}$$
$$= \sum_{j=1}^{q_v}\frac{\underline{\omega}_j(k,l)}{\lambda^j_{kl}}\left(\ln(\mathrm{e}^{\vec{\lambda}_{kl}(\vec{P}+\vec{V})}-1) - \ln\left(\mathrm{e}^{\vec{\lambda}_{kl}\vec{P}}-1\right)\right)$$

with the factor $\displaystyle\sum_{j=1}^{q_v}\frac{\underline{\omega}_j(k,l)}{\lambda^j_{kl}} = \underline{\vec{b}}(k,l)\vec{b}^{-1}(k,l)$

$$\times\left(\frac{\underline{a}(k,l)-a(k,l)}{a(k,l)-1} + q\right)^{-1}$$
$$:= \underline{\alpha}_v, \quad (154)$$

providing

$$\int\sum_{j=1}^{q_v}\mathrm{d}V_j\underline{c}_{js}\begin{bmatrix} s \\ kl \end{bmatrix} = \underline{\alpha}_v\ \ln\left(\frac{\mathrm{e}^{\vec{\lambda}_{kl}(\vec{P}+\vec{V})}-1}{\mathrm{e}^{\vec{\lambda}_{kl}\vec{P}}-1}\right)$$
$$= \ln\left(\frac{\mathrm{e}^{\vec{\lambda}_{kl}\vec{V}} - \mathrm{e}^{-\vec{\lambda}_{kl}\vec{P}}}{1-\mathrm{e}^{-\vec{\lambda}_{kl}\vec{P}}}\right)^{\underline{\alpha}_v} \quad (155)$$

Inserting this expression into reference (54), the general result for classes of condensations of reference (55) is obtained.

We finally want to give the concrete expressions for the possible classes of condensations according to the definition of the hermetry forms in subsection 3.4.3:

In the hermetry form a only the partial structure 1 with the coordinates x_5 and x_6 is hermetric, so we have $\vec{V} = \vec{x}_5 + \vec{x}_6 := i\vec{T}$, $\vec{P} = 0$ and only one condenser function according to (45)

$$F_{(1)}(a) = \mathrm{e}^{\mathrm{i}\underline{\vec{\lambda}}\vec{T}}\left(\mathrm{e}^{\mathrm{i}\vec{\lambda}\vec{T}}-1\right)^{-\underline{\alpha}}$$
$$= \left(\mathrm{e}^{\mathrm{i}(\vec{\lambda}-\underline{\vec{\lambda}}\underline{\alpha}^{-1})\vec{T}} - \mathrm{e}^{-\mathrm{i}\underline{\vec{\lambda}}\underline{\alpha}^{-1}\vec{T}}\right)^{-\underline{\alpha}}$$
$$= \left(\mathrm{e}^{\mathrm{i}\vec{\lambda}(1-f)\vec{T}} - \mathrm{e}^{-\mathrm{i}\vec{\lambda}f\vec{T}}\right)^{-\underline{\alpha}}$$
$$\to \left(1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{T}}\right)^{-\underline{\alpha}} \quad \text{if} \quad f\to 1 \quad (156)$$

where we have used, as in (56), $\underline{\vec{\lambda}}\underline{\alpha}^{-1}\vec{V} = \vec{\lambda}\vec{V}f$. To keep the notation simple, we have suppressed the coordinate indices k,l at $\underline{\vec{\lambda}}$, $\vec{\lambda}$, $\underline{\alpha}$ and $f = f(k,l)$.

For the hermetry form b there are three possible combinations of the hermetric coordinates $\vec{x}_4 = ict\,\vec{e}_4 := i\vec{ct}$ and $\vec{x}_5 + \vec{x}_6 := i\vec{T}$ which yield (with (56), respectively (45))

$$F_{(1)}(b) = \left(\frac{\mathrm{e}^{\mathrm{i}\vec{\lambda}(1-f)\vec{T}} - \mathrm{e}^{-\mathrm{i}\vec{\lambda}(\vec{ct}+f\vec{T})}}{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{ct}}}\right)^{-\underline{\alpha}}$$
$$\to \left(\frac{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}(\vec{ct}+\vec{T})}}{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{ct}}}\right)^{-\underline{\alpha}}$$
$$F_{(2)}(b) = \left(\frac{\mathrm{e}^{\mathrm{i}\vec{\lambda}(1-f)\vec{ct}} - \mathrm{e}^{-\mathrm{i}\vec{\lambda}(\vec{T}+f\vec{ct})}}{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{T}}}\right)^{-\underline{\alpha}} \quad (157)$$
$$\to \left(\frac{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}(\vec{ct}+\vec{T})}}{1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{T}}}\right)^{-\underline{\alpha}}$$
$$F_{(12)}(b) = \left(\mathrm{e}^{\mathrm{i}\vec{\lambda}(1-f)(\vec{ct}+\vec{T})} - \mathrm{e}^{-\mathrm{i}\vec{\lambda}f(\vec{ct}+\vec{T})}\right)^{-\underline{\alpha}}$$
$$\to \left(1-\mathrm{e}^{-\mathrm{i}\vec{\lambda}(\vec{ct}+\vec{T})}\right)^{-\underline{\alpha}}$$

including the results for the limit $f\to 1$.

For the form c with the hermetric coordinates $\vec{r} = \sum_{k=1}^{3}x_k\vec{e}_k$ and again $\vec{x}_5 + \vec{x}_6 = i\vec{T}$, which describes neutral ponderable particles as complex-valued space condensations, there is only one class with a leading imaginary form ($F_{(1)}(c)$) and the class according to (45):

$$F_{(1)}(c) = \left(\frac{e^{i\vec{\lambda}(1-f)\vec{T}} - e^{-\vec{\lambda}(\vec{r}+if\vec{T})}}{1-e^{-\vec{\lambda}\vec{r}}}\right)^{-\underline{\alpha}}$$

$$\to \left(\frac{1-e^{-\vec{\lambda}(\vec{r}+i\vec{T})}}{1-e^{-\vec{\lambda}\vec{r}}}\right)^{-\underline{\alpha}} \tag{158}$$

$$F_{(13)}(c) = \left(e^{\vec{\lambda}(1-f)(\vec{r}+i\vec{T})} - e^{-\vec{\lambda}f(\vec{r}+i\vec{T})}\right)^{-\underline{\alpha}}$$

$$\to \left(1-e^{-\vec{\lambda}(\vec{r}+i\vec{T})}\right)^{-\underline{\alpha}}$$

The hermetry form d with the hermetric coordinates $\vec{r}$, $i\vec{ct}$ and $i\vec{T}$, describing charged ponderable particles as complex-valued space condensations, provides six classes:

$$F_\alpha(d) = \left(\frac{e^{\vec{\lambda}(1-f)\vec{V}_\alpha} - e^{-\vec{\lambda}(\vec{P}_\alpha+f\vec{V}_\alpha)}}{1-e^{-\vec{\lambda}\vec{P}_\alpha}}\right)^{-\underline{\alpha}}$$

$$\to \left(\frac{1-e^{-\vec{\lambda}(\vec{P}_\alpha+\vec{V}_\alpha)}}{1-e^{-\vec{\lambda}\vec{P}_\alpha}}\right)^{-\underline{\alpha}} \quad \text{with}$$

$$\vec{V}_\alpha = i\vec{T}, \vec{P}_\alpha = \vec{r} + i\vec{ct} \quad \text{for} \quad \alpha = (1)$$

$$\vec{V}_\alpha = i\vec{ct}, \vec{P}_\alpha = \vec{r} + i\vec{T} \quad \text{for} \quad \alpha = (2)$$

$$\vec{V}_\alpha = i\vec{T} + i\vec{ct}, \vec{P}_\alpha = \vec{r} \quad \text{for} \quad \alpha = (12)$$

$$\vec{V}_\alpha = \vec{r} + i\vec{T}, \vec{P}_\alpha = i\vec{ct} \quad \text{for} \quad \alpha = (13)$$

$$\vec{V}_\alpha = \vec{r} + i\vec{ct}, \vec{P}_\alpha = i\vec{T} \quad \text{for} \quad \alpha = (23) \qquad \text{and}$$

$$F_{(123)}(d) = \left(e^{\vec{\lambda}(1-f)(\vec{r}+i\vec{ct}+i\vec{T})} - e^{-\vec{\lambda}f(\vec{r}+i\vec{ct}+i\vec{T})}\right)^{-\underline{\alpha}}$$

$$\to \left(1-e^{-\vec{\lambda}(\vec{r}+i\vec{ct}+i\vec{T})}\right)^{-\underline{\alpha}} \tag{159}$$

These results correspond to those of Heim in [44], page 186 (his reference 80), but are slightly different, as Heim evaluates $\vec{\underline{\lambda}} = \underline{\alpha}\,\vec{\lambda}$ exactly, which we do not, compare reference (140) above. Only if the function $f \to 1$, the results become identical to Heim's. Note that, although we have abstained here from marking them with indices, the exponent $\underline{\alpha}$ and the function f in general differ between the single results, as according to their general expressions (136), (143) and (144), they depend on the particular hermetric coordinate structure of each class.

Finally, we analyse the condenser functions of the form as given in reference (55) concerning their dependency on the variable $\vec{P}$.[59] We are interested in the functions $F_{(13)}(d)$ and $F_{(23)}(d)$ in which one of the variables $\vec{P} = i\vec{ct}$ or $\vec{P} = i\vec{T}$, in the following generalised as $\vec{P} = i\vec{z}$, appears in the term $e^{-\vec{\lambda}\vec{P}}$, and the other variable is part of the vector $\vec{V} = \vec{r} + i\vec{\xi} - i\vec{z}$:

$$F_{(1/23)}(d) = e^{\,\vec{\underline{\lambda}}(\vec{r}+i\vec{\xi}-i\vec{z})}\left(\frac{e^{\,\vec{\lambda}(\vec{r}+i\vec{\xi}-i\vec{z})} - e^{-i\vec{\lambda}\vec{z}}}{1-e^{-i\vec{\lambda}\vec{z}}}\right)^{-\underline{\alpha}}$$

$$= e^{\,\vec{\underline{\lambda}}(\vec{r}+i\vec{\xi})}\left(e^{\,\vec{\lambda}(\vec{r}+i\vec{\xi})} - 1\right)^{-\underline{\alpha}} e^{-i\vec{\underline{\lambda}}\vec{z}}\, e^{\,i\underline{\alpha}\vec{\lambda}\vec{z}}$$

$$\times \left(1-e^{-i\vec{\lambda}\vec{z}}\right)^{\underline{\alpha}}$$

$$= F_{(123)}(d)\; e^{-i\vec{\underline{\lambda}}\vec{z}}\left(e^{i\vec{\lambda}\vec{z}} - 1\right)^{\underline{\alpha}} \tag{160}$$

Here the definition of $F_{(123)}(d)$ in its initial form (45) was used. We continue with the abbreviation $x = \vec{\lambda}\vec{z}$ and the definition $\vec{\underline{\lambda}}\vec{z} = \underline{\alpha}\, f_z \vec{\lambda}\vec{z}$ in analogy to (144) (i.e. f_z depends on the indices k, l and in general on $\vec{z}$ as well) and use the substitution $e^{i\vec{\lambda}\vec{z}} - 1 = \cos x - 1 + i\sin x := re^{i\phi}$, meaning $r^2 = (\cos x - 1)^2 + \sin^2 x$, $\phi = \arctan(\frac{\sin x}{\cos x - 1})$ and $(re^{i\phi})^{\underline{\alpha}} = r^{\underline{\alpha}} e^{i\phi\underline{\alpha}} = (2 - 2\cos x)^{\underline{\alpha}/2}(\cos \underline{\alpha}\phi + i\sin\underline{\alpha}\phi)$:

$$F_{(1/23)}(d) = F_{(123)}(d)\,(\cos\underline{\alpha}\, f_z x - i\sin\underline{\alpha}\, f_z x)$$

$$\times (\cos\underline{\alpha}\phi + i\sin\underline{\alpha}\phi)(2(1-\cos x))^{\underline{\alpha}/2}$$

$$= F_{(123)}(d)\,(\cos\underline{\alpha}(f_z x - \phi) - i\sin\underline{\alpha}(f_z x - \phi))$$

$$\times (2(1-\cos x))^{\underline{\alpha}/2} \tag{161}$$

We now assume that f_z hardly varies with $\vec{z}$, i.e. x, and at the (for our purposes) relevant index pair $k, l = 4$ takes a numerical value similar to $f_{ext}(4,4)$ (see Section 4). Then the dependence on the variable $\vec{z}$ is defined by x and ϕ (see above) and is periodic in form of the cos-, respectively, sin-function, and $(2(1-\cos x))^{\underline{\alpha}/2}$ is a modulating factor. In the area of data given in Section 4 (for f_{ext} and thus $\underline{\alpha}$) this factor only creates small peaks at the $\cos x = 1$ points, but does not change the overall numerical result that the arithmetic mean of $F_{(1/23)}(d)$ over a whole period (depending on $\vec{z}$) is essentially zero.

Appendix I: Simplification of equation (64)

To simplify the term in the square bracket of (64) it is set $= Re^{i\phi}$ and the abbreviation $K = K(n)$ used so that $R^2 = (e^{Kd}\cos K - 1)^2 + e^{2Kd}\sin^2 K = e^{2Kd} - 2e^{Kd}\cos K + 1 = e^{2Kd} + 1$ for $m = 1$ and $= e^{2Kd} \pm 2e^{Kd} + 1 = (e^{Kd} \pm 1)^2$ for $m = 2$. For both $R \to e^{Kd}$ holds if $e^{Kd} \gg 1$, which is true in our data range. The angle ϕ is given by $\phi = \arctan(\frac{e^{Kd}\sin K}{e^{Kd}\cos K - 1})$ and $(Re^{i\phi})^{-\underline{\alpha}} = R^{-\underline{\alpha}}(\cos \underline{\alpha}\phi - i\sin\underline{\alpha}\phi) \to e^{-\underline{\alpha}Kd}(\cos\underline{\alpha}\phi - i\sin\underline{\alpha}\phi) = e^{-\frac{\pi}{2}\underline{n}d\, f_{ext}^{-1}}(\cos$

59 The subsequent results were not derived by Heim.

$\left(\frac{\pi n}{2K} f_{\text{ext}}^{-1}\phi\right) - \mathrm{i}\sin\left(\frac{\pi n}{2K} f_{ext}^{-1}\phi\right)\big)$. The angle ϕ can be evaluated in the range $e^{Kd} \gg 1$ via $\cos K = \cos(\frac{\pi}{2}(2n+1)) = 0$ and $\sin K = \sin(\frac{\pi}{2}(2n+1)) = (-1)^n$ for $m = 1$ and $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ for $m = 2$. This gives $\phi = \arctan((-1)^{n+1}\mathrm{e}^{\frac{\pi}{2}(2n+1)d}) \to (-1)^{n+1}\frac{\pi}{2}$ for $m = 1$ and $n > 0$ and $\phi = \arctan(0) = 0$ for $m = 2$.

With these results the expressions of reference (65) are easily obtained.

Appendix J: Linear approximation, Dirac and Maxwell equations

We study the linear limit of the fundamental equations of Heim's theory (according to [44], chapter VIII.5, pages 358–363). We can expect that in case of the b hermetry, a linear approximation should lead to the known equations which determine the electromagnetic field and the particles interacting with this field, i.e. the equations of Maxwell in the classical limit and Dirac's equation for a relativistic fermion field (particle).

The b hermetry contains 6 possible signatures in terms of combinations of partial structures, namely the elements (1), (11), (2), (22), (12) and (21), which can occur in the co- and the contravariant signatures. If we now consider Eq. (43), neglect the non-linear terms in it and assume a stationary status in which only the components of the R_4 are non-zero and all field components and also eigenvalues λ_i for the trans-dimensions x_5, x_6 become zero, and finally sum over two indices $F_m = \sum_k F_{km}^k$, we get

$$\partial_m F_p - \partial_p F_m = \underline{\lambda}_p F_m := i\lambda_p F_m \tag{162}$$

where we have re-defined the $\underline{\lambda}_p$. The relevant partial structures of the b hermetry lead to two different condenser functions according to Appendix H if we take into account the limit $x_5, x_6 = 0$, thus $\vec{T} = 0$, namely the functions $F_{(1)}(b)$ and $F_{(12)}(b)$ according to (157). This means that there is a second equation for the second function, which we name G:

$$\partial_m G_p - \partial_p G_m = \underline{\lambda}_p G_m := \mathrm{i}\lambda_p G_m \tag{163}$$

(Heim uses the notation of functions g_m and h_m, see [44] page 359 bottom.) Summation over $1 \leq p \leq 4$ leads to

$$\begin{aligned} &\sum_p \partial_m F_p - \partial_p F_m = \mathrm{i}\lambda F_m \quad \text{and} \\ &\sum_p \partial_m G_p - \partial_p G_m = \mathrm{i}\lambda G_m \end{aligned} \tag{164}$$

with $\sum_p \lambda_p = \lambda$, or in matrix representation

$$\begin{pmatrix} \partial_1 - \sum_p \partial_p & \partial_1 & \partial_1 & \partial_1 \\ \partial_2 & \partial_2 - \sum_p \partial_p & \partial_2 & \partial_2 \\ \partial_3 & \partial_3 & \partial_3 - \sum_p \partial_p & \partial_3 \\ \partial_4 & \partial_4 & \partial_4 & \partial_4 - \sum_p \partial_p \end{pmatrix} \vec{F}(\vec{G}) = \mathrm{i}\lambda \vec{F}(\vec{G}). \tag{165}$$

These two linear matrix equations for $\vec{F}$ and $\vec{G}$ can be transformed to one matrix equation with two other quadruple vectors $\vec{a}$ and $\vec{b}$ which are linearly coupled:[60]

$$\begin{pmatrix} \pm i\partial_4 & 0 & \partial_3 & \partial_1 & 0 & 0 & 0 & \partial_2 \\ 0 & \pm i\partial_4 & \partial_1 & -\partial_3 & 0 & 0 & -\partial_2 & 0 \\ \partial_3 & \partial_1 & \pm i\partial_4 & 0 & 0 & \partial_2 & 0 & 0 \\ \partial_1 & -\partial_3 & 0 & \pm i\partial_4 & -\partial_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & \pm i\partial_4 & 0 & \partial_3 & \partial_1 \\ 0 & 0 & \partial_2 & 0 & 0 & \pm i\partial_4 & \partial_1 & -\partial_3 \\ 0 & -\partial_2 & 0 & 0 & \partial_3 & \partial_1 & \pm i\partial_4 & 0 \\ \partial_2 & 0 & 0 & 0 & \partial_1 & -\partial_3 & 0 & \pm i\partial_4 \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \mathrm{i}\lambda \begin{pmatrix} -a_1 \\ -a_2 \\ a_3 \\ a_4 \\ -b_1 \\ -b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{166}$$

This system can be written as a four-dimensional equation of a complex vector field $\vec{\phi} = \vec{a} + i\vec{b}$

$$\begin{pmatrix} \pm i\partial_4 & 0 & \partial_3 & \partial_1 - \mathrm{i}\partial_2 \\ 0 & \pm i\partial_4 & \partial_1 + \mathrm{i}\partial_2 & -\partial_3 \\ \partial_3 & \partial_1 - \mathrm{i}\partial_2 & \pm i\partial_4 & 0 \\ \partial_1 + \mathrm{i}\partial_2 & -\partial_3 & 0 & \pm i\partial_4 \end{pmatrix} \vec{\phi} = \mathrm{i}\lambda \begin{pmatrix} -\phi_1 \\ -\phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \tag{167}$$

which in turn can be expressed through the well-known four-dimensional matrices of the Dirac equation

$$\left(\pm \mathrm{i}\partial_4 + \partial_1 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \right.$$

60 In (166), we have corrected the sign errors which [44] obviously contains at this point.

$$+\partial_3\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\Bigg)\vec{\phi}$$

$$= -\mathrm{i}\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\vec{\phi} \tag{168}$$

or compact (and with partial derivatives according to x^k written out)

$$\left(\pm\mathrm{i}\frac{\partial}{\partial x^4}+\alpha_1\frac{\partial}{\partial x^1}+\alpha_2\frac{\partial}{\partial x^2}+\alpha_3\frac{\partial}{\partial x^3}\right)\vec{\phi}=-\mathrm{i}\lambda\beta\,\vec{\phi}\ \text{ with}$$
$$\alpha_i=\begin{pmatrix}0&\sigma_i\\\sigma_i&0\end{pmatrix},\quad i=1,2,3,\quad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{169}$$

the σ_i being the Pauli matrices. The α_i, β fulfil the anti-commutator relations $[\alpha_i,\alpha_k]=2\delta_{ik}, [\alpha_i,\beta]=0$ and $\alpha_i^2=\beta^2=1$. The eigenvalues λ have the dimension meter^{-1} and should be proportional to the mass of an electric charge carrier if the photon field of the b hermetry interacts with the field of this charge carrier. On the other hand it is always a matter of quantum structure so that λ can be regarded as ratio of the momentum mc to $\hbar$, thus $\lambda = mc/\hbar$. With $x_4 = ict$ this gives

$$\pm i\hbar\frac{\partial}{\partial t}\vec{\phi}=\frac{\hbar c}{i}\sum_{k=1}^{3}\alpha_k\frac{\partial}{\partial x^k}\vec{\phi}+mc^2\beta\,\vec{\phi} \tag{170}$$

which is identical to the Dirac equation if the four-dimensional complex vector field $\vec{\phi}$ is equalised with a four-dimensional spinor ψ and the branch with negative sign in front of the energy operator $\hbar\frac{\partial}{\partial t}$ is omitted.[61] The usual form of the equation which explicitly shows the Lorentz invariance can be obtained by multiplying with the matrix β/c on both sides and introducing the gamma matrices $\gamma^0=\beta, \gamma^i=\beta\alpha_i, i=1,2,3$ and $\mu=0,1,2,3$ [76]:

$$\left(\mathrm{i}\hbar\gamma^\mu\frac{\partial}{\partial x^\mu}-mc\right)\psi=0 \tag{171}$$

Hence, it is possible to obtain the Dirac equation, the linear wave equation of relativistic quantum mechanics, from Heim's theory (basic equation (43)) as a *possible* result in the linear limit of hermetry b. However, only its structure as linear partial differential equation of first order, acting on a complex 4-dimensional function, can be deduced unambiguously, but not its Clifford algebra, given by the γ matrices. Equation (165) does not provide them automatically, but allows only a linear transformation to them.

We now consider the macroscopic domain (i.e. Heim's fourth area of validity) and an empty R_3 space which means mass $m\to 0$, thus $\lambda\to 0$. Going back to the system of equations (166) and setting the components of the vectors $\vec{a}$ and $\vec{b}$ as $a_1\sim E_3, a_2\sim E_1, a_3=0, a_4\sim -B_2, b_1=0, b_2\sim E_2, b_3\sim B_3, b_4\sim B_1$, we obtain the set of equations

$$\begin{aligned}
\pm\mathrm{i}\partial_4E_3-\partial_1B_2+\partial_2B_1&=0,\\
\pm\mathrm{i}\partial_4E_1+\partial_3B_2-\partial_2B_3&=0,\\
\partial_3E_3+\partial_1E_1+\partial_2E_2&=0,\\
\partial_1E_3-\partial_3E_1-\pm\mathrm{i}\partial_4B_2&=0,\\
\partial_2B_2+\partial_3B_3+\partial_1B_1&=0,\\
\pm\mathrm{i}\partial_4E_2+\partial_1B_3-\partial_3B_1&=0,\\
-\partial_2E_1+\partial_1E_2+\pm\mathrm{i}\partial_4B_3&=0,\\
\partial_2E_3-\partial_3E_2+\pm\mathrm{i}\partial_4B_1&=0
\end{aligned} \tag{172}$$

which, when using again $x_4 = ict$, can be written in the compact form

$$\begin{aligned}
\mathrm{rot}\vec{B}&=\pm\frac{1}{c}\frac{\partial\vec{E}}{\partial t}, &\mathrm{div}\vec{E}&=0,\\
\mathrm{rot}\vec{E}&=-\pm\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, &\mathrm{div}\vec{B}&=0
\end{aligned} \tag{173}$$

and obviously corresponds to the microscopic Maxwell equations in the vacuum. An exact identity is obtained with the positive branch of the $\pm$ signs and when transforming $\vec{B}\to c\vec{B}$.

So, Heim's theory allows connecting to the classical theory of electromagnetism on a mathematical level.

Appendix K: The spin in Heim's theory

We have seen in subsection 3.5.1 and Appendix H, subsection H.1 that a spin field tensor can be derived from the properties of the g_{ik} and $\left[{i \atop kl}\right]$, meaning that in a hermetric space V_r always $g_{-ik}:=P_{ik}=(\mathrm{rot}_{(x)}\vec{\phi})_{ik}=\mathrm{const}\neq 0$ holds for the spin field tensor P_{ik}, with $\vec{\phi}$ being a so-called spin field vector.

We now want to connect these quantities to the properties of a spin in 3-dimensional space as an angular momentum-like quantity (see chapter VII.3 in [44]). In

61 Solutions with negative energy appear when solving Eq. (171) anyway and are interpreted as solutions for the anti-particle e^+.

analogy to the electromagnetic field we construct the angular momentum density from the energy momentum tensor T^{kl} as

$$L^{ikl} = \frac{1}{w}(x^i T^{kl} - x^k T^{il}) \tag{174}$$

with w being the absolute value of the imaginary part of the 6-dimensional world velocity ($\vec{Y} = \vec{v} + i\vec{w}, w^2 = c^2 + \dot{\epsilon}^2 + \dot{\eta}^2$). The divergence of L becomes

$$\begin{aligned}\partial_l L^{ikl} &= \frac{1}{w}\left[\left(\partial_l x^i\right) T^{kl} + x^i\left(\partial_l T^{kl}\right) - \left(\partial_l x^k\right) T^{il}\right. \\ &\quad \left. - x^k\left(\partial_l T^{il}\right)\right] \\ &= \frac{1}{w}\left(\delta^i_l T^{kl} - \delta^k_l T^{il}\right) \\ &= \frac{1}{w}(T^{ki} - T^{ik}) = \frac{1}{w}(T^{ki} - (T^{ki})^\times) \\ &= \frac{1}{w}(T^{ki})^-, \end{aligned} \tag{175}$$

having used that the divergence of T vanishes. For a hermitian T, i.e. $T^- = 0$, this means that the angular momentum L is conserved. Vice versa a conservation of angular momentum forces $T^- = 0$, which with (18) and (28) leads to $\widehat{\{\}} = \widehat{\{\}}^\times$ and $g = g^\times$. In the metronised poly-metric case, despite $g_{(\kappa\lambda)} \neq g^\times_{(\kappa\lambda)}$, it remains $\widehat{\begin{bmatrix}\kappa\lambda\\ \mu\nu\end{bmatrix}}_- = 0$ (see (42)) which leads to the constant metron spin $g^{(\kappa\lambda)}_- := P^{(\kappa\lambda)} = \text{const}$, compare above.

A spatial spin density vector in 6 dimensions is now defined by a trace over the angular momentum density tensor

$$\sigma^i = \frac{1}{w}\sum_j \varepsilon_{jkl} x^k T^{li} = \frac{1}{w}\sum_j x^k \varepsilon_{klj} T^{li} := \frac{\vec{x}}{w}\cdot \vec{\tilde{P}}^i \tag{176}$$

with the Levi-Civita symbol ε_{jkl} and sums over identical indices. The tensor $\tilde{P}^{ki} = \sum_j \varepsilon_{klj} T^{li}$ is in general not identical to P^{ki} of above, but structurally similar. It has the dimension of an energy density, so it can be written as a sum of energy density contributions $\tilde{P} = \sum_\alpha W_\alpha$ where the W_α depend on the partial geometry of the respective hermetry form, denoted by the usual indices, and can be expressed as $W_\alpha = W^{(\kappa\lambda)}_{(\mu\nu)} \sim \text{Tr}\, C\widehat{\begin{bmatrix}\kappa\lambda\\ \mu\nu\end{bmatrix}}_+ = \vec{\lambda}^{(\kappa\lambda)}_{(\mu\nu)}\widehat{\begin{bmatrix}\kappa\lambda\\ \mu\nu\end{bmatrix}}_+$ according to (57). If we now integrate over the 3-dimensional space in the metron calculus (compare Section 4) we get a spin vector

$$\begin{aligned} s^i &= \int \partial V \sigma^i = \int \partial V \frac{\vec{x}}{w}\cdot\vec{\tilde{P}}^i = \int \partial V \frac{x_k}{w}\sum_\alpha W^{ki}_\alpha \\ &= \sum_\alpha \int \partial E^{ki}_\alpha \frac{x_k}{w} \end{aligned} \tag{177}$$

where i denotes the index of the vector and α stands for the hermetric partial structure. For the differential ∂E^{ki}_α the quantisation of energy must be taken into account $E = h\nu$ so that the differential becomes $\partial E^{ki}_\alpha = h\nu^k \partial m^{ki}_\alpha$, with m^{ki}_α being selectors of whole quantum numbers when the metron calculus is applied.[62] This provides

$$s^i = h\sum_\alpha \int \partial m^{ki}_\alpha \frac{x_k}{w}\nu^k. \tag{178}$$

In Heim's model the velocity w and the frequencies ν^k can be related, as the spin is considered as a result of the cyclical flux of condensations (compare Subsection 3.5.3). The world vector then lies on a circular path, $\vec{x} = \vec{e}_x \frac{u}{2\pi}$, with the perimeter u, in Cartesian coordinates $x_k = \vec{e}_k \cdot \vec{e}_x \frac{u}{2\pi} := \frac{u_k}{2\pi}$. Heim now relates $w = 2u_l\nu^l$ which means that the circulation velocities $u_l\nu^l$ (perimeter times frequency) shall be determined by w. The factor 2 obviously corresponds to the property of the quantum mechanical spin that the spin state vector is identically reproduced only after a rotation of a 4π angle. After inserting in (178) we obtain

$$s^i = h\sum_\alpha \int \partial m^{ki}_\alpha \frac{u_k\nu^k}{4\pi u_l \nu^l} \quad\rightarrow\quad \frac{\hbar}{2}\sum_\alpha \int \partial m^i_\alpha = \frac{\hbar}{2} m^i \tag{179}$$

if the selectors $m^{ki}_\alpha \rightarrow m^i_\alpha$ do not depend on index k, i.e. simply form a vector, and $m^i = \sum_\alpha m^i_\alpha$, being natural numbers. Thus we find the well-known quantitative result of the quantum mechanical spin.

We can write the result as a vector and take into account that in the 6-dimensional space the sector of the coordinates $x_4 - x_6$ is imaginary:

$$\vec{s} = \frac{\hbar}{2}\sum_{i=1}^{6} m^i \vec{e}_i := \hbar(\vec{\sigma}_r + \mathrm{i}\vec{\sigma}_t) \tag{180}$$

The real part $\hbar\vec{\sigma}_r$ must be the spin in 3-dimensional space, the second imaginary part is identified by Heim as the isospin.

Appendix L: Qualitative derivation of an interaction potential

(*Consideration of the author, not by Heim*)

In Sections 3 and 4, we have derived state functions for partial geometries and calculated mass-energies for related particle states which are associated with the hermetry forms c and d in Heim's theory. An important question is also, how the interaction between the particles can be described, which must be related to the strong interaction.

62 Note that the ν^k form a vector, not a 2-dimensional tensor.

We can use the fundamental geodesic equation (12) to derive an ansatz for an interaction potential:

In the nonrelativistic limit for velocities $\ll c$ we may neglect $\mathrm{d}\vec{x}/\mathrm{d}\tau$ (with the proper time $\mathrm{d}\tau$) with respect to $\mathrm{d}t/d\tau$ and write (12) as[63]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i_{44}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{44} = 0. \tag{181}$$

If we now define $\Gamma^i_{44} = -\frac{\partial}{\partial x_i}\Phi$, we get the classical equation of motion $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{\partial}{\partial x_i}\Phi$ with the potential $\Phi(\vec{x})$. This means that a nonrelativistic potential can be determined from the relation

$$\frac{\partial}{\partial x_i}\Phi(\vec{x}) = -\Gamma^i_{44} \;\rightarrow\; -F^i_{44} \tag{182}$$

when the classical Christoffel symbol is transformed to Heim's poly-metric state function F, as defined in subsection 3.5. We are interested in the course of the Φ in the 3-dimensional space and consider therefore only the F^i_{44} at their extrema with respect to the remaining coordinates $\vec{\xi}$. Hence, we can use the expressions of (51), (52) and (159) for $F_{(3)\mathrm{ext}}$ and $F_{(123)\mathrm{ext}}$. To these expressions we apply the same approximation as in (53), which is not only valid in the limit of big r, which was considered in (53), but also for middle to small r if $\underline{\alpha}_{44} = (\underline{\beta}_{\pm}/\beta_{\pm}) f^{-1}_{\mathrm{ext}}$ is determined by the value of $f_{\mathrm{ext}} \approx -2.16$, as specified in Subsection 4.3 (i.e. r in the same order of magnitude as given by $|f_{\mathrm{ext}}|$). The terms in front of the e-function are constant (C^i) with regard to $\vec{r}$, and so we get

$$\begin{aligned}\frac{\partial}{\partial x_i}\Phi(\vec{r}) &= -C^i\, \mathrm{e}^{-\underline{\alpha}_{44}\vec{\lambda}\vec{r}\,(1-f(\vec{r}))}\\ &= -C^i\, \mathrm{e}^{-\underline{\alpha}_{44}\lambda r\,\cos\,\theta\,(1-f(\vec{r}))}\end{aligned} \tag{183}$$

with θ as angle between the $\vec{\lambda}$ and $\vec{r}$. Now we have to model the regarding its exact course unknown function $f(\vec{r})$, respectively, $1 - f(\vec{r})$. First of all, we assume a radial symmetry, i.e. $f = f(r)$. Next, we know that $1 - f$ has to be approximately constant at a value of $1 - f_{\mathrm{ext}} \approx 3.16$ for small r and that it has to change sign in the course to bigger r, see Subsection 4.2. This can be modelled by the approach

$$\begin{aligned}1 - f(r) &= a_0 \approx 3.16 \quad \text{for} \quad r \le r_0,\\ &= -br + c \quad \text{for} \quad r_0 < r < r_1,\\ &= -a_1 < 0 \quad \text{for} \quad r \ge r_1\end{aligned} \tag{184}$$

63 See [52]. As in the whole paper, we again use the index 4 for the time coordinate.

with $r_1 \gg r_0$ and requiring that the function is steady which defines the constants $b > 0$ and $c > 0$ in dependence of the other parameters. To come to a simply assessable relation, we make a restriction and hide the dependency on θ, so consider only that on r. Inserting $1 - f(r)$ as defined by (184) and assuming $C^i < 0$, we obtain a function on the r.h.s. of (185) with the course of a shifted Gaussian function

$$\begin{aligned}\frac{\partial}{\partial r}\Phi(r) &\sim \mathrm{e}^{Ar\,(1-f(r))}\\ &\rightarrow \mathrm{e}^{-A(br^2-cr)} \sim \mathrm{e}^{-Ab\left(r-\frac{c}{2b}\right)^2} \quad \text{for } r_0 < r < r_1\end{aligned} \tag{185}$$

with the abbreviation $A = -\underline{\alpha}_{44}\lambda\cos\theta$. A is > 0, as $f^{-1}_{\mathrm{ext}} < 0$ ($\cos\theta > 0$ assumed). Its width depends via b, c on the unknown parameters r_0, r_1. With this function, Eq. (185) cannot be integrated analytically, but the course of $\Phi(r)$ can be detected approximately. A Taylor expansion of the Gaussian up to sixth order and subsequent integration yields a result which can be well approximated by a linear function $\Phi(r) \approx kr + \tilde{c}$ in the area $r_0 < r < c/b$. The term c/b marks the zero of $1 - f(r) = -br + c$. For $r \ge c/b$ Φ flattens out and approximates a constant value c_1. In the sectors $r \le r_0$ and $r \ge r_1$ the function $1 - f$ is constant in our model, so here (185) can easily be integrated and Φ be determined as $\Phi(r) \sim \frac{\mathrm{e}^{Aa_0 r}}{Aa_0} + c_0$, respectively, $\sim -\frac{\mathrm{e}^{-Aa_1 r}}{Aa_1} + c_1$. The integration constants $\tilde{c}, c_0, c_1$ have to be set so that $\Phi(r)$ is steady.

We conclude that we have derived a potential from the hermetry forms c, d in Heim's theory by means of the resulting state functions and from an *a priori* unknown function f, modelled in a way that empirical data are met – the fit of f in the inner region so that the mass spectrum comes out, and its necessary behaviour for $r \to \infty$. We obtain a result with an approximately linearly increasing course in a region which we assume as decisive for the question whether free particles can "escape" at higher energies and in which particle shape. It is interesting to notice that we find this linear course $\sim kr + \tilde{c}$ which is also the typical potential for the confinement in the quark model. As we currently cannot determine the quantities r_0, r_1 and the further introduced constants, we cannot make any quantitative statements, but consider our qualitative result as encouragement for further research.

Appendix M: Heim's minimum particle mass

Here we shortly outline Heim's approach to derive a smallest particle mass, which turns out to lie very close to the electron mass. We cannot reproduce his entire derivation due to the limited space of this paper and refer to [43], chapter II.4 and chapter IV.3 pages 226–248, and to [44], chapter V.1, for the complete description, but present some essential parts.

M.1 Gravitational space structures and their extrema

From his approach to take into account the field mass in the overall energy–mass balance of gravity (compare Section 3.1) Heim derives a non-linear differential equation (see page 85 in [43]) with an implicit solution for the gravitational potential $\phi(r)$

$$rq\mathrm{e}^{-q} = A(1-r/\rho)^2 \qquad \text{with} \qquad q = 1-\sqrt{1-\varepsilon\phi},$$
$$M_0 = Lm_0, \quad \varepsilon = \frac{3}{8c^2} \tag{186}$$

which can be calculated approximately after expansion of $q\mathrm{e}^{-q} \approx \varepsilon\phi/2$ to first order and fixing of the integration constant A so that the empirical Newtonian law is reproduced for $r \ll \rho$:

$$A = \frac{\varepsilon\gamma M_0}{2}, \quad \phi(r) \rightarrow \frac{\gamma Lm(r)}{r}(1-r/\rho)^2 \tag{187}$$

γ is the gravitational constant, M_0 the macroscopic field source, consisting of L microscopic elementary masses m_0. This means $\phi > 0$ and accelleration $g < 0$ for $r < \rho$, i.e. an attractive gravitational field which is limited at $r = \rho$ with $\phi(\rho) = 0$ and $g(\rho) = 0$. For $r > \rho$ in contrast, $\phi > 0$ increases again, and with $g > 0$ a weak repelling field appears. To determine the distance ρ one realises that ρ should be related to a quantum wavelength $\lambda' = w/\nu_w$ with $\epsilon_g = h\nu_w$, as $\lambda' = 2\rho$ if we consider $L = 1$, thus only an atomic mass unit m_0 of M_0. An approximation in a Taylor series of (186) ($q\mathrm{e}^{-q}$ to second order) gives $(1-r/\rho)^2 \sim rq\mathrm{e}^{-q} \approx \frac{1}{2}r\varepsilon\phi(1-\varepsilon\phi/2)$ or $m(r)(1-\varepsilon\phi/2) =$ const due to (187), thus $m(r)(1-\varepsilon\phi(r)/2) = m_0(1-\varepsilon\phi(r_0)/2)$ or $m(r) - m_0 = m(r)\varepsilon\phi(r)/2 - m_0\varepsilon\phi(r_0)/2$. With $\phi(\rho) = 0$ this leads to a field mass of $\mu_g = m(\rho) - m_0 = -\varepsilon m_0\phi(r_0)/2 = -\frac{\varepsilon m_0^2\gamma}{2r_0}(1-r_0/\rho)^2$. On the other hand $\mu_g = -h/(2w\rho)$ follows from $\epsilon_g = -\mu_g w^2 = h\nu_w = \frac{hw}{2\rho}$. If the length $2r_0$ is identified with the Compton wavelength of the energy m_0c^2, $\lambda = \frac{h}{m_0c} = 2r_0$, the equation $\rho(1-\frac{h}{2m_0c\rho})^2 = \frac{h^2}{\gamma m_0^3}$ is obtained[64] and thereby the approximate solution

$$\rho = \frac{h^2}{\gamma m_0^3} \tag{188}$$

for sufficiently weak fields.[65] Reference (186)–(187) is not yet a structure-theoretical relation, but a phenomenological extension of the basic laws d1, d2 and c.

Now one can derive extrema of the space structure (186) with help of the fact that the r.h.s. of (186) is always real, $\mathrm{Im}(q\mathrm{e}^{-q}) = 0 \rightarrow 1-\varepsilon\phi \geq 0$. Is $r = R > \rho$ the position of this border of reality, this means $\phi(R) = \phi_{ext} = 1/\varepsilon$ or $\varepsilon\gamma M(R) = R(1-R/\rho)^{-2}$. On the other hand $q(R) = 1$ holds (from (186)) which provides $\varepsilon\gamma M(R) = eA$ and thus $M(R) = M_0e/2$. This leads to a quadratic equation for $x = R/\rho$:

$(1-x)^2 = \frac{2\rho}{e\varepsilon\gamma M_0}x$. With the abbreviation $\alpha = \frac{\rho}{e\varepsilon\gamma M_0} = \frac{8c^2\rho}{3e\gamma M_0}$ the two solutions appear

$$R_\pm = \rho(\alpha+1)(1 \pm \sqrt{1-(\alpha+1)^{-2}}),$$
$$\text{for} \quad \alpha \gg 1 \rightarrow \rho\alpha(1 \pm \sqrt{1-\alpha^{-2}}) \rightarrow \rho\alpha(1 \pm (1-1/(2\alpha^2))). \tag{189}$$

The last convergence is valid if all terms $(1/\alpha^2)^\nu \approx 0$ for $\nu > 1$ which always is fulfilled approximately. Then, the very simplified approximation can be derived

$$R_+ = 2\alpha\rho, \quad R_- = \frac{\rho}{2\alpha}, \quad R_+R_- = \rho^2 \tag{190}$$

from which $R_+ < \infty$ follows. With α an obviously microscopic value is obtained for the other limit

$$R_- = \frac{e}{2}\varepsilon\gamma M_0 = \frac{3e}{16}\frac{\gamma M_0}{c^2} \tag{191}$$

which is largely identical to the Schwarzschild radius of general relativity, which in turn is a measure for the extension of a black hole.

Now the existence of only one single mass system $M_0 = Lm_0$ in the R_3 is considered, with $L = 1$ being reduced to one atomic element with mass $m_M(R)$. This gives

$$R_- = \rho_M(\alpha_M+1)(1-\sqrt{1-(\alpha_M+1)^{-2}}) \tag{192}$$

64 Here, Heim uses his result $w = \frac{4}{3}c$ for the propagation velocity of gravitational disturbances or waves. This result $\neq c$ can be disputed, but was obtained by the same geometrical spherical approach the term ε results from. In the obtained equation the factors cancel out.

65 Heim calculates the distance ρ for a "realistic" microscopic $m_0 = A_t m_N$ with m_N the nucleon mass and an atomic weight A_t of 1.9–2.4 which corresponds to the empirical Russell composition of intergalactic matter, and obtains a value between 10^7 and $2\cdot10^7$ light years. Heim concludes that this result could explain the non-existence of higher-order systems in the cosmos, because if the distance between the components of a galaxy cluster lies below this limit, then these components must be attracted by each other and optically show a picture of the order of such systems, while at distances of $r > \rho$ and $g > 0$ matter is distributed completely chaotically.

where α_M is the corresponding value of α with $m_M(R) = M_0 e/2$. According to (187), R_- fulfills the reality border $1 - \varepsilon\phi(R_-) = 0$ as per $R_-/\varepsilon = \gamma m_M(R_-)(1 - R_-/\rho_M)^2$. With the Compton wavelength $\lambda = h/(m_M c)$ this becomes $\lambda R_- = \frac{\gamma h}{2wc^2}(1 - R_-/\rho_M)^2$ (expressions for ε and w used). For empty space $\lambda \to \infty$ the limits $\lim_{\lambda\to\infty} m_M(R_-) = \lim_{\lambda\to\infty} 1/\rho_M = \lim_{\lambda\to\infty} R_- = 0$ are reached (due to (188) and (191)), for $\lim_{\lambda\to\infty} 1/\alpha_M \sim \lim_{\lambda\to\infty} \frac{R_-}{\rho_M} = 0$, too. With these results the limit of the improper product λR_- can be determined as well:

$$\tau = \lim_{\lambda\to\infty}(\lambda R_-) = \frac{\gamma h}{2wc^2} = \frac{3\gamma h}{8c^3} = \text{const} > 0 \qquad (193)$$

τ obviously is a universal constant with the dimension of an area that only depends on other fundamental physical constants which determine the laws of nature d1, d2 and c. This constant, which Heim calls "*metron*" and which we already introduced in Section 3.3, (24), is obviously independent of whether the space is empty or not. Since the relation (15) cannot be executed to an infinitesimal R_4-element, the term (193) must be extended to the R_4, but to the R_6 as well, as such differences also arise for R_6-volumes due to Eq. (28).

The results above serve Heim as ingredients to derive further cosmological theorems which lie beyond the scope of this work and can be found in [44].

M.2 Mass formula from elementary hermetric structures

The next step towards a calculation of a minimum mass consists of an analysis of the structural properties of the c hermetry. In subsection 3.4.5 we found the relation (36) between the coordinates r and ξ for the respective eigenvalue spectra. $\xi^2 = \epsilon^2 + \eta^2$ and squaring Eq. (36) yields $r^2 - (\frac{2n_r+1}{2n_\xi+1})^2\epsilon^2 = (\frac{2n_r+1}{2n_\xi+1})^2\eta^2$ (n_ξ is identical to n_- of (36)). The two quantum numbers describe the course of the metronic eigenvalues in the range $0 \le r^2 < \xi^2$ where $n_r < n_\xi$ must hold. For $r = \epsilon$, i.e. on the singular area in case of $\eta \ne 0$, η/r becomes $\eta/r = \pm\sqrt{(\frac{2n_\xi+1}{2n_r+1})^2 - 1} = \pm 2(2n_r+1)^{-1}\sqrt{(n_\xi + n_r + 1)(n_\xi - n_r)}$. With the definitions $n_\xi = n$ and $n_r = n_\xi - j$ we get $\eta/r = \pm 2(2n+1-2j)^{-1}\sqrt{j(2n+1-j)}$ and for $j_{\min} = 1$ which provides the closest approximation $n_r \to n_\xi$ and hence minimal value for η/r, $\eta/r = \pm 2(2n-1)^{-1}\sqrt{2n} := \pm 2f(n)$.

Heim's strategy is now to find a relation of two fundamental lengths which could be equated with η/r and thereby define a mass spectrum. He considers the characteristic distances $R_-, \lambda_C/2, \rho$ and R_+. $\lambda_C = h/(mc)$ is the Compton wavelength of a matter quantum. It can be seen that only the relations $R_-/(\lambda_C/2), R_-/\rho$ and $\lambda_C/(2\rho)$ could serve for this purpose, amongst others as the relation $|\eta/r| \le 2\sqrt{2}$ must be fulfilled. Since the mass terms lie in the subatomic range, the approximations of (188) and (189) can be used, (189) in the approximate form $\rho = 2(1+\alpha)R_-$. With (188) and the definition $\mu = \sqrt{ch/\gamma}$ the term $\alpha = \frac{8c^2\rho}{3e\gamma m}$ can be written as $\alpha = \frac{8}{3e}(\mu/m)^4$. Inserting supplies $R_-/(\lambda_C/2) = (\mu/m)^2 (1+\alpha)^{-1} = (m/\mu)^2((m/\mu)^4 + \frac{8}{3e})^{-1} = 2f(n)$, $R_-/\rho = \frac{1}{2}(1+\alpha)^{-1} = \frac{1}{2}(m/\mu)^4((m/\mu)^4 + \frac{8}{3e})^{-1} = 2f(n)$ and $\lambda_C/(2\rho) = \frac{1}{2}(m/\mu)^2 = 2f(n)$.

Solving the first two equations for the mass m shows that m becomes complex ($\mathrm{Im}(m) \ne 0$) for $1 \le n \le 8$, which contradicts physical reality. The third solution $(m/\mu)^2 = 4f(n)$ in contrast remains real for all $n > 0$ and $m = 0$ for $n = 0$. This means that this solution is the only one which provides a physically relevant expression for the mass. The mass spectrum for neutral quanta of the c hermetry, which implicitly includes all possible gravitational and photonic condensations and therefore cannot yet serve as a discrete partial spectrum of matter particles, thus is to be set as

$$m(n) = 2\sqrt{\frac{ch}{\gamma}}\,\frac{\sqrt[4]{2n}}{\sqrt{2n-1}}. \qquad (194)$$

Next Heim analyses how the obtained result must be expanded for the d hermetry, which has the same complex coordinate structure $\vec{x} = \vec{r} + i\vec{\xi}$, but with $\vec{\xi}$ containing also the time coordinate ct. We learned in Subsection 3.4.5 that the Hermetry form d describes charged particles. Thus the influence of the coupling of the electromagnetic field on the energy and hence mass terms has to be developed. After a somewhat longer heuristic derivation, which as another result provides a formula for the fine structure constant,[66] Heim obtains the result

$$m(n,q) = \frac{1}{2}m(n)\,\eta_q, \quad \eta_q = \frac{\pi}{\sqrt[4]{4q^4 + \pi^4}} \qquad (195)$$

where q is the electric charge quantum number. Note that Heim adjusted the result by a factor 1/2 also for neutral masses ($q = 0$). From (194) can be seen that there is an upper limit $m_{\max} = \sqrt{ch/\gamma}\sqrt[4]{2}\,\eta_q$ (for $n = 1$) which is (regarding the order of magnitude) identical to the *maximons* once conceived in particle physics [77].[67]

66 $\alpha = \frac{9}{(2\pi)^5}\vartheta$, $\vartheta = 5\eta + 2\sqrt{\eta} + 1$, $\eta = \eta_1$.

67 which were not necessarily supposed to be real particles. Here the actual hypothetical maximon mass $\sqrt{ch/\gamma}$ only appears as a gauge factor.

M.3 The minimal mass

In chapter V.1 “The minimal complex condensation” of [44] Heim derives a minimal mass for matter quanta, i.e. for the c and d hermetries, based on the results described in the two previous subsections. Starting from the relations (190), one can derive $\rho^2 = eR_+A$, with A defined as in (187) which can also be expressed through the metron and the Compton wavelengh as $A = \frac{\tau}{2\lambda}$. The last two expressions combined give $eR_+\tau = 2\lambda\rho^2$.

Referring to formula (194), Heim now reasons that a minimal mass is reached for a maximum number of metrons $n = n_N$ by which the distance R_+ is represented. In the metron picture he makes the ansatz that the volume $2\lambda\rho^2$ is projected onto the meridian plane of the matter quantum m by a grid selector C which operates on the number of metrons $Cn = p\sqrt{\tau}n$. Here p is a projective factor.[68] Then the field source is expressed by this metron number n along the distance R_+: $eR_+\tau = 2\lambda\rho^2 = FCn = pF\sqrt{\tau}n$ where F is the unit area that, according to a projection of all spatial potential areas of the field structure in the plane R_2, is limited by a contour line: $F = \pi s_0^2$ for spherical potential areas and $p^2\sqrt{F} = s_0 = 1[m]$, thus $p = \pi^{-1/4}$.[69] Then

$$e\sqrt{\tau}R_+ = pFn = \pi^{3/4}\, n \tag{196}$$

results. From this relation and from (188), which is an optimal approximation for a minimum mass, $n = \frac{16\ h^2\mu^4}{3\pi^{3/4}\gamma m^7}\sqrt{\tau}$ can be obtained. For this minimum and thus a maximum $n = n_N$, (194) and (195) can be approximated as $\left(\frac{m}{\mu\eta_q}\right)^4 = \frac{2n_N}{(2n_N-1)^2} \approx \frac{1}{2n_N}$, hence $n_N \approx \frac{1}{2}\left(\frac{\mu\eta_q}{m}\right)^4$. Equating both expressions for n, n_N yields $\frac{32\ h^2}{3\pi^{3/4}\gamma m^3}\sqrt{\tau} = \eta_q^4$ and thus, with the expression (193) for τ, the minimum mass results as

$$m = \frac{4\sqrt[4]{\pi}}{c\,\eta_q\sqrt[3]{\eta_q}}\sqrt[3]{3\pi\gamma\hbar}\sqrt{\frac{c\hbar}{3\gamma}}. \tag{197}$$

Inserting the fundamental physical constants $c, \hbar$ and γ gives $m_1 = 0.5137$ MeV for charge number $q = 1$, which is close to the mass of the electron. For $q = 0$ formula (197) yields $m_0 = 0.5069$ MeV. Finally, Heim considers the value $m_{(0)} = m_1/\eta_1 = 0.5189$ MeV which can be considered as neutral complement of m_1 according to (195). In the frame of his full mass spectrum theory (which we do not follow in this paper due to the reasons mentioned in Subsection 4.1), Heim derives a more complex formula which reproduces the electron mass even better, and he associates m_0 with a speculative neutral electron which, however, has never been found in experiments so far (but cannot be easily detected either).

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68 We use p instead of Heim’s notation ε to avoid confusion with the other factor ε already introduced above.

69 Phrasing of [55] used and translated. The relation $p^2\sqrt{F} = s_0$ is not clear to the author.

Appendix N: Complete calculated mass spectrum

Table 2: Calculation of rest energies (masses) in MeV for scenario 1, i.e. same logic as in Table 1: Even and odd numbers of mass units ($m = 1, 2$) allowed for both bosons and fermions.

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
Leptons:													
e	0.511	1	0	1	1	0.511	0.00	0.511	0.00	0.511	0.00	0.511	0.00
μ	105.66	1	2	3	3	106.26	0.57	106.43	0.73	104.51	1.09	105.04	0.59
τ	1776.86	1	2	51	51	1779.65	0.16	1778.60	0.10	1776.70	0.01	1785.65	0.49
Quarks:													
u, d	315.00**	1	2	9	9	316.16	0.37	315.44	0.14	313.54	0.46	315.11	0.04
s	525.00**	1	2	15	15	525.34	0.07	524.46	0.10	522.56	0.46	525.19	0.04
c	1270.00	2	2	18	36	1254.14	1.25	1254.14	1.25	1254.14	1.25	1260.46	0.75
b	4180.00	2	2	60	120	4180.47	0.01	4180.47	0.01	4180.47	0.01	4201.52	0.51
t	172,760.0	2	4	36	72	171,001.4	1.02	171,001.4	1.02	171,001.4	1.02	172,728.0	0.02
Mesons:													
unflavoured:													
$\pi^{\pm}$	139.57	2	2	2	4	139.35	0.16	139.35	0.16	139.35	0.16	140.05	0.34
π^0	134.98	2	2	2	4	139.35	3.24	139.35	3.24	139.35	3.24	140.05	3.76
η	547.86	2	2	8	16	557.40	1.74	557.40	1.74	557.40	1.74	560.20	2.25
$f_0(500)$	400–550	1	2	13	13	455.63		454.79		452.88		455.17	
$\rho(770)$	775.26	2	2	11	22	772.51	0.36	770.22	0.65	766.42	1.14	770.28	0.64
$\omega(782)$	782.65	2	2	11	22	772.51	1.30	770.22	1.59	766.42	2.07	770.28	1.58
$\eta'(958)$	957.78	1	2	27	27	943.50	1.49	942.51	1.59	940.61	1.79	945.34	1.30
$f_0(980)$	990.00	2	2	14	28	975.44	1.47	975.44	1.47	975.44	1.47	980.36	0.97
$a_0(980)$	980.00	2	2	14	28	975.44	0.46	975.44	0.46	975.44	0.46	980.36	0.04
$\phi(1020)$	1019.46	1	2	29	29	1013.18	0.62	1012.18	0.71	1010.28	0.90	1015.37	0.40
$h_1(1170)$	1166.00	1	2	33	33	1152.54	1.15	1151.53	1.24	1149.63	1.40	1155.42	0.91
$b_1(1235)$	1229.50	1	2	35	35	1222.22	0.59	1221.20	0.67	1219.30	0.83	1225.44	0.33
$a_1(1260)$	1230.00	1	2	35	35	1222.22	0.63	1221.20	0.72	1219.30	0.87	1225.44	0.37
$f_2(1270)$	1275.50	1	2	37	37	1291.90	1.29	1290.88	1.21	1288.98	1.06	1295.47	1.57
$f_1(1285)$	1281.90	1	2	37	37	1291.90	0.78	1290.88	0.70	1288.98	0.55	1295.47	1.06
$\eta(1295)$	1294.00	1	2	37	37	1291.90	0.16	1290.88	0.24	1288.98	0.39	1295.47	0.11
$\pi(1300)$	1300.00	1	2	37	37	1291.90	0.62	1290.88	0.70	1288.98	0.85	1295.47	0.35
$a_2(1320)$	1316.90	2	2	19	38	1329.88	0.99	1327.62	0.81	1323.82	0.53	1330.48	1.03
$f_0(1370)$	1200–1500	1	2	39	39	1361.58		1360.55		1358.65		1365.50	
$\pi_1(1400)$	1354.00	1	2	39	39	1361.58	0.56	1360.55	0.48	1358.65	0.34	1365.50	0.85
$\eta(1405)$	1408.80	2	2	20	40	1393.49	1.09	1393.49	1.09	1393.49	1.09	1400.51	0.59
$h_1(1415)$	1416.00	1	2	41	41	1431.26	1.08	1430.23	1.00	1428.33	0.87	1435.52	1.38
$a_1(1420)$	1411.00	2	2	20	40	1393.49	1.24	1393.49	1.24	1393.49	1.24	1400.51	0.74

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$f_1(1420)$	1426.30	1	2	41	41	1431.26	0.35	1430.23	0.28	1428.33	0.14	1435.52	0.65
$\omega(1420)$	1410.00	2	2	20	40	1393.49	1.17	1393.49	1.17	1393.49	1.17	1400.51	0.67
$f_2(1430)$	1430.00	1	2	41	41	1431.26	0.09	1430.23	0.02	1428.33	0.12	1435.52	0.39
$a_0(1450)$	1474.00	2	2	21	42	1469.23	0.32	1466.96	0.48	1463.17	0.74	1470.53	0.24
$\rho(1450)$	1465.00	2	2	21	42	1469.23	0.29	1466.96	0.13	1463.17	0.13	1470.53	0.38
$\eta(1475)$	1475.00	2	2	21	42	1469.23	0.39	1466.96	0.54	1463.17	0.80	1470.53	0.30
$f_0(1500)$	1506.00	1	2	43	43	1500.94	0.34	1499.90	0.40	1498.00	0.53	1505.55	0.03
$f_1(1510)$	1518.00	2	2	22	44	1532.84	0.98	1532.84	0.98	1532.84	0.98	1540.56	1.49
$f_2'(1525)$	1517.40	1	2	43	43	1500.94	1.08	1499.90	1.15	1498.00	1.28	1505.55	0.78
$f_2(1565)$	1542.00	2	2	22	44	1532.84	0.59	1532.84	0.59	1532.84	0.59	1540.56	0.09
$\rho(1570)$	1570.00	1	2	45	45	1570.62	0.04	1569.58	0.03	1567.68	0.15	1575.57	0.35
$h_1(1595)$	1594.00	2	2	23	46	1608.57	0.91	1606.31	0.77	1602.51	0.53	1610.58	1.04
$\pi_1(1600)$	1660.00	2	2	24	48	1672.19	0.73	1672.19	0.73	1672.19	0.73	1680.61	1.24
$a_1(1640)$	1655.00	1	2	47	47	1640.30	0.89	1639.25	0.95	1637.35	1.07	1645.60	0.57
$f_2(1640)$	1639.00	1	2	47	47	1640.30	0.08	1639.25	0.02	1637.35	0.10	1645.60	0.40
$\eta_2(1645)$	1617.00	2	2	23	46	1608.57	0.52	1606.31	0.66	1602.51	0.90	1610.58	0.40
$\omega(1650)$	1670.00	2	2	24	48	1672.19	0.13	1672.19	0.13	1672.19	0.13	1680.61	0.64
$\omega_3(1670)$	1667.00	2	2	24	48	1672.19	0.31	1672.19	0.31	1672.19	0.31	1680.61	0.82
$\pi_2(1670)$	1670.60	2	2	24	48	1672.19	0.10	1672.19	0.10	1672.19	0.10	1680.61	0.60
$\phi(1680)$	1680.00	2	2	24	48	1672.19	0.46	1672.19	0.46	1672.19	0.46	1680.61	0.04
$\rho_3(1690)$	1688.80	2	2	24	48	1672.19	0.98	1672.19	0.98	1672.19	0.98	1680.61	0.48
$\rho(1700)$	1720.00	1	2	49	49	1709.97	0.58	1708.92	0.64	1707.03	0.75	1715.62	0.25
$a_2(1700)$	1705.00	1	2	49	49	1709.97	0.29	1708.92	0.23	1707.03	0.12	1715.62	0.62
$f_0(1710)$	1704.00	1	2	49	49	1709.97	0.35	1708.92	0.29	1707.03	0.18	1715.62	0.68
$\eta(1760)$	1751.00	2	2	25	50	1747.92	0.18	1745.66	0.30	1741.86	0.52	1750.63	0.02
$\pi(1800)$	1810.00	2	2	26	52	1811.54	0.08	1811.54	0.08	1811.54	0.08	1820.66	0.59
$f_2(1810)$	1815.00	2	2	26	52	1811.54	0.19	1811.54	0.19	1811.54	0.19	1820.66	0.31
X(1835)	1826.50	2	2	26	52	1811.54	0.82	1811.54	0.82	1811.54	0.82	1820.66	0.32
$\phi_3(1850)$	1854.00	1	2	53	53	1849.33	0.25	1848.27	0.31	1846.38	0.41	1855.67	0.09
$\eta_2(1870)$	1842.00	1	2	53	53	1849.33	0.40	1848.27	0.34	1846.38	0.24	1855.67	0.74
$\pi_2(1880)$	1874.00	2	2	27	54	1887.27	0.71	1885.01	0.59	1881.21	0.38	1890.69	0.89
$\rho(1900)$	1900.00	2	2	27	54	1887.27	0.67	1885.01	0.79	1881.21	0.99	1890.69	0.49
$f_2(1910)$	1900.00	2	2	27	54	1887.27	0.67	1885.01	0.79	1881.21	0.99	1890.69	0.49
$a_0(1950)$	1931.00	1	2	55	55	1919.00	0.62	1917.95	0.68	1916.05	0.77	1925.70	0.27
$f_2(1950)$	1936.00	2	2	28	56	1950.89	0.77	1950.89	0.77	1950.89	0.77	1960.71	1.28
$a_4(1970)$	1967.00	2	2	28	56	1950.89	0.82	1950.89	0.82	1950.89	0.82	1960.71	0.32
$\rho_3(1990)$	1990.00	1	2	57	57	1988.68	0.07	1987.62	0.12	1985.72	0.21	1995.72	0.29
$\pi_2(2005)$	1963.00	2	2	28	56	1950.89	0.62	1950.89	0.62	1950.89	0.62	1960.71	0.12
$f_2(2010)$	2011.00	2	2	29	58	2026.62	0.78	2024.36	0.66	2020.56	0.48	2030.74	0.98
$f_0(2020)$	1992.00	1	2	57	57	1988.68	0.17	1987.62	0.22	1985.72	0.32	1995.72	0.19

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\pi_2(2100)$	2090.00	2	2	30	60	2090.24	0.01	2090.24	0.01	2090.24	0.01	2100.76	0.51
$f_0(2100)$	2086.00	2	2	30	60	2090.24	0.20	2090.24	0.20	2090.24	0.20	2100.76	0.71
$f_2(2150)$	2157.00	2	2	31	62	2165.96	0.42	2163.71	0.31	2159.91	0.13	2170.79	0.64
$\rho(2150)$	2150.00	2	2	31	62	2165.96	0.74	2163.71	0.64	2159.91	0.46	2170.79	0.97
$\phi(2170)$	2160.00	2	2	31	62	2165.96	0.28	2163.71	0.17	2159.91	0.00	2170.79	0.50
$f_0(2200)$	2187.00	1	2	63	63	2197.71	0.49	2196.65	0.44	2194.75	0.35	2205.80	0.86
$f_J(2220)$	2231.10	2	2	32	64	2229.59	0.07	2229.59	0.07	2229.59	0.07	2240.81	0.44
$\eta(2225)$	2221.00	2	2	32	64	2229.59	0.39	2229.59	0.39	2229.59	0.39	2240.81	0.89
$\rho_3(2250)$	2250.00	1	2	65	65	2267.39	0.77	2266.32	0.73	2264.42	0.64	2275.83	1.15
$f_2(2300)$	2297.00	2	2	33	66	2305.31	0.36	2303.06	0.26	2299.26	0.10	2310.84	0.60
$f_4(2300)$	2300.00	2	2	33	66	2305.31	0.23	2303.06	0.13	2299.26	0.03	2310.84	0.47
$f_0(2330)$	2330.00	1	2	67	67	2337.06	0.30	2336.00	0.26	2334.10	0.18	2345.85	0.68
$f_2(2340)$	2345.00	1	2	67	67	2337.06	0.34	2336.00	0.38	2334.10	0.46	2345.85	0.04
$\rho_5(2350)$	2330.00	1	2	67	67	2337.06	0.30	2336.00	0.26	2334.10	0.18	2345.85	0.68
$f_6(2510)$	2465.00	1	2	71	71	2476.41	0.46	2475.34	0.42	2473.45	0.34	2485.90	0.85
strange:													
$K^{\pm}$	493.68	2	2	7	14	493.84	0.03	491.53	0.43	487.72	1.21	490.18	0.71
K^0	497.61	2	2	7	14	493.84	0.76	491.53	1.22	487.72	1.99	490.18	1.49
$K_0^*(700)$	630–730	2	2	10	20	696.75		696.75		696.75		700.25	
$K^*(892)$	894.24	2	2	13	26	911.85	1.97	909.57	1.71	905.77	1.29	910.33	1.80
$K_1(1270)$	1253.00	2	2	18	36	1254.14	0.09	1254.14	0.09	1254.14	0.09	1260.46	0.60
$K_1(1400)$	1403.00	2	2	20	40	1393.49	0.68	1393.49	0.68	1393.49	0.68	1400.51	0.18
$K^*(1410)$	1414.00	1	2	41	41	1431.26	1.22	1430.23	1.15	1428.33	1.01	1435.52	1.52
$K_0^*(1430)$	1425.00	1	2	41	41	1431.26	0.44	1430.23	0.37	1428.33	0.23	1435.52	0.74
$K_2^*(1430)$	1429.85	1	2	41	41	1431.26	0.10	1430.23	0.03	1428.33	0.11	1435.52	0.40
K(1460)	1460.00	2	2	21	42	1469.23	0.63	1466.96	0.48	1463.17	0.22	1470.53	0.72
$K_2(1580)$	1580.00	1	2	45	45	1570.62	0.59	1569.58	0.66	1567.68	0.78	1575.57	0.28
K(1630)	1629.00	1	2	47	47	1640.30	0.69	1639.25	0.63	1637.35	0.51	1645.60	1.02
$K_1(1650)$	1672.00	2	2	24	48	1672.19	0.01	1672.19	0.01	1672.19	0.01	1680.61	0.51
$K^*(1680)$	1718.00	1	2	49	49	1709.97	0.47	1708.92	0.53	1707.03	0.64	1715.62	0.14
$K_2(1770)$	1773.00	1	2	51	51	1779.65	0.38	1778.60	0.32	1776.70	0.21	1785.65	0.71
$K_3^*(1780)$	1776.00	1	2	51	51	1779.65	0.21	1778.60	0.15	1776.70	0.04	1785.65	0.54
$K_2(1820)$	1819.00	2	2	26	52	1811.54	0.41	1811.54	0.41	1811.54	0.41	1820.66	0.09
K(1830)	1874.00	2	2	27	54	1887.27	0.71	1885.01	0.59	1881.21	0.38	1890.69	0.89
$K_0^*(1950)$	1945.00	2	2	28	56	1950.89	0.30	1950.89	0.30	1950.89	0.30	1960.71	0.81
$K_2^*(1980)$	1943.00	2	2	28	56	1950.89	0.41	1950.89	0.41	1950.89	0.41	1960.71	0.91
$K_4^*(2045)$	2048.00	1	2	59	59	2058.36	0.51	2057.30	0.45	2055.40	0.36	2065.75	0.87
$K_2(2250)$	2247.00	2	2	32	64	2229.59	0.78	2229.59	0.78	2229.59	0.78	2240.81	0.28
$K_3(2320)$	2324.00	1	2	67	67	2337.06	0.56	2336.00	0.52	2334.10	0.43	2345.85	0.94

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$K_5^\star(2380)$	2382.00	2	2	34	68	2368.93	0.55	2368.93	0.55	2368.93	0.55	2380.86	0.05
$K_4(2500)$	2490.00	1	2	71	71	2476.41	0.55	2475.34	0.59	2473.45	0.66	2485.90	0.16
K(3100)	3100.00	1	2	89	89	3103.50	0.11	3102.41	0.08	3100.52	0.02	3116.13	0.52
charmed:													
$D^\pm$	1869.65	2	2	27	54	1887.27	0.94	1885.01	0.82	1881.21	0.62	1890.69	1.13
D^0	1864.83	1	2	53	53	1849.33	0.83	1848.27	0.89	1846.38	0.99	1855.67	0.49
$D^\star(2007)^0$	2006.85	1	2	57	57	1988.68	0.91	1987.62	0.96	1985.72	1.05	1995.72	0.55
$D^\star(2010)^\pm$	2010.26	2	2	29	58	2026.62	0.81	2024.36	0.70	2020.56	0.51	2030.74	1.02
$D_0^\star(2300)^0$	2300.00	2	2	33	66	2305.31	0.23	2303.06	0.13	2299.26	0.03	2310.84	0.47
$D_0^\star(2300)^\pm$	2349.00	1	2	67	67	2337.06	0.51	2336.00	0.55	2334.10	0.63	2345.85	0.13
$D_1(2420)^0$	2420.50	1	2	69	69	2406.74	0.57	2405.67	0.61	2403.77	0.69	2415.88	0.19
$D_1(2420)^\pm$	2423.20	1	2	69	69	2406.74	0.68	2405.67	0.72	2403.77	0.80	2415.88	0.30
$D_1(2430)^0$	2427.00	2	2	35	70	2444.66	0.73	2442.41	0.63	2438.61	0.48	2450.89	0.98
$D_2^\star(2460)^0$	2460.56	2	2	35	70	2444.66	0.65	2442.41	0.74	2438.61	0.89	2450.89	0.39
$D_2^\star(2460)^\pm$	2465.40	1	2	71	71	2476.41	0.45	2475.34	0.40	2473.45	0.33	2485.90	0.83
$D(2550)^0$	2564.00	1	2	73	73	2546.09	0.70	2545.02	0.74	2543.12	0.81	2555.93	0.31
$D_J^\star(2600)$	2623.00	1	2	75	75	2615.77	0.28	2614.69	0.32	2612.80	0.39	2625.95	0.11
$D^\star(2640)^\pm$	2637.00	2	2	38	76	2647.63	0.40	2647.63	0.40	2647.63	0.40	2660.97	0.91
$D(2740)^0$	2737.00	2	2	39	78	2723.36	0.50	2721.10	0.58	2717.31	0.72	2730.99	0.22
$D_3^\star(2750)$	2763.50	1	2	79	79	2755.12	0.30	2754.04	0.34	2752.14	0.41	2766.00	0.09
$D(3000)^0$	3214.00	2	2	46	92	3205.03	0.28	3205.03	0.28	3205.03	0.28	3221.17	0.22
charmed, strange:													
$D_s^\pm$	1968.34	2	2	28	56	1950.89	0.89	1950.89	0.89	1950.89	0.89	1960.71	0.39
$D_s^{\star\pm}$	2112.20	1	2	61	61	2128.03	0.75	2126.97	0.70	2125.07	0.61	2135.77	1.12
$D_{s0}^\star(2317)^\pm$	2317.80	2	2	33	66	2305.31	0.54	2303.06	0.64	2299.26	0.80	2310.84	0.30
$D_{s1}(2460)^\pm$	2459.50	2	2	35	70	2444.66	0.60	2442.41	0.69	2438.61	0.85	2450.89	0.35
$D_{s1}(2536)^\pm$	2535.11	1	2	73	73	2546.09	0.43	2545.02	0.39	2543.12	0.32	2555.93	0.82
$D_{s2}^\star(2573)$	2569.10	2	2	37	74	2584.01	0.58	2581.76	0.49	2577.96	0.34	2590.94	0.85
$D_{s1}^\star(2700)^\pm$	2708.30	2	2	39	78	2723.36	0.56	2721.10	0.47	2717.31	0.33	2730.99	0.84
$D_{s1}^\star(2860)^\pm$	2859.00	2	2	41	82	2862.70	0.13	2860.45	0.05	2856.66	0.08	2871.04	0.42
$D_{s3}^\star(2860)^\pm$	2860.50	2	2	41	82	2862.70	0.08	2860.45	0.00	2856.66	0.13	2871.04	0.37
$D_{sJ}(3040)^\pm$	3044.00	1	2	87	87	3033.82	0.33	3032.74	0.37	3030.84	0.43	3046.10	0.07
bottom:													
$B^\pm$	5279.34	1	2	151	151	5263.42	0.30	5262.33	0.32	5260.43	0.36	5286.92	0.14
B^0	5279.65	1	2	151	151	5263.42	0.31	5262.33	0.33	5260.43	0.36	5286.92	0.14
$B^\star$	5324.70	1	2	153	153	5333.10	0.16	5332.00	0.14	5330.10	0.10	5356.94	0.61
$B_1(5721)^+$	5725.90	2	2	82	164	5713.31	0.22	5713.31	0.22	5713.31	0.22	5742.08	0.28
$B_1(5721)^0$	5726.10	2	2	82	164	5713.31	0.22	5713.31	0.22	5713.31	0.22	5742.08	0.28
$B_J^\star(5732)$	5698.00	1	2	163	163	5681.47	0.29	5680.37	0.31	5678.47	0.34	5707.07	0.16

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$B_2^\star(5747)^+$	5737.20	1	2	165	165	5751.15	0.24	5750.05	0.22	5748.15	0.19	5777.10	0.70
$B_2^\star(5747)^0$	5739.50	1	2	165	165	5751.15	0.20	5750.05	0.18	5748.15	0.15	5777.10	0.66
$B_J(5840)^+$	5851.00	2	2	84	168	5852.66	0.03	5852.66	0.03	5852.66	0.03	5882.13	0.53
$B_J(5840)^0$	5863.00	2	2	84	168	5852.66	0.18	5852.66	0.18	5852.66	0.18	5882.13	0.33
$B_J(5970)^+$	5964.00	1	2	171	171	5960.17	0.06	5959.07	0.08	5957.17	0.11	5987.17	0.39
$B_J(5970)^0$	5971.00	1	2	171	171	5960.17	0.18	5959.07	0.20	5957.17	0.23	5987.17	0.27
bottom, strange:													
B_s^0	5366.88	2	2	77	154	5370.98	0.08	5368.74	0.03	5364.94	0.04	5391.96	0.47
$B_s^\star$	5415.40	1	2	155	155	5402.77	0.23	5401.67	0.25	5399.78	0.29	5426.97	0.21
$X(5568)^\pm$	5566.90	2	2	80	160	5573.96	0.13	5573.96	0.13	5573.96	0.13	5602.03	0.63
$B_{s1}(5830)^0$	5828.70	1	2	167	167	5820.82	0.14	5819.72	0.15	5817.82	0.19	5847.12	0.32
$B_{s2}^\star(5840)^0$	5839.86	2	2	84	168	5852.66	0.22	5852.66	0.22	5852.66	0.22	5882.13	0.72
$B_{sJ}^\star(5850)$	5853.00	2	2	84	168	5852.66	0.01	5852.66	0.01	5852.66	0.01	5882.13	0.50
bottom, charmed:													
B_c^+	6274.90	2	2	90	180	6270.71	0.07	6270.71	0.07	6270.71	0.07	6302.29	0.44
$B_c(2S)^\pm$	6871.60	1	2	197	197	6865.94	0.08	6864.84	0.10	6862.94	0.13	6897.50	0.38
$c\bar{c}$:***													
$\eta_c(1S)$	2983.90	2	2	43	86	3002.05	0.61	2999.80	0.53	2996.01	0.41	3011.09	0.91
$J/\psi(1S)$	3096.90	1	2	89	89	3103.50	0.21	3102.41	0.18	3100.52	0.12	3116.13	0.62
$\chi_{c0}(1P)$	3414.71	2	2	49	98	3420.10	0.16	3417.85	0.09	3414.05	0.02	3431.24	0.48
$\chi_{c1}(1P)$	3510.67	1	2	101	101	3521.55	0.31	3520.46	0.28	3518.56	0.22	3536.28	0.73
$h_c(1P)$	3525.38	1	2	101	101	3521.55	0.11	3520.46	0.14	3518.56	0.19	3536.28	0.31
$\chi_{c2}(1P)$	3556.17	2	2	51	102	3559.45	0.09	3557.20	0.03	3553.40	0.08	3571.30	0.43
$\eta_c(2S)$	3637.50	2	2	52	104	3623.08	0.40	3623.08	0.40	3623.08	0.40	3641.32	0.11
$\psi(2S)$	3686.10	2	2	53	106	3698.79	0.34	3696.55	0.28	3692.75	0.18	3711.35	0.68
$\psi(3770)$	3773.70	2	2	54	108	3762.42	0.30	3762.42	0.30	3762.42	0.30	3781.37	0.20
$\psi_2(3823)$	3822.20	2	2	55	110	3838.14	0.42	3835.90	0.36	3832.10	0.26	3851.40	0.76
$\psi_3(3842)$	3842.71	2	2	55	110	3838.14	0.12	3835.90	0.18	3832.10	0.28	3851.40	0.23
$\chi_{c0}(3860)$	3862.00	1	2	111	111	3869.92	0.21	3868.83	0.18	3866.94	0.13	3886.41	0.63
$\chi_{c1}(3872)$	3871.69	1	2	111	111	3869.92	0.05	3868.83	0.07	3866.94	0.12	3886.41	0.38
$Z_c(3900)$	3888.40	2	2	56	112	3901.77	0.34	3901.77	0.34	3901.77	0.34	3921.42	0.85
X(3915)	3918.40	2	2	56	112	3901.77	0.42	3901.77	0.42	3901.77	0.42	3921.42	0.08
$\chi_{c2}(3930)$	3922.20	1	2	113	113	3939.60	0.44	3938.51	0.42	3936.61	0.37	3956.43	0.87
X(3940)	3942.00	1	2	113	113	3939.60	0.06	3938.51	0.09	3936.61	0.14	3956.43	0.37
$X(4020)^\pm$	4024.10	1	2	115	115	4009.27	0.37	4008.18	0.40	4006.29	0.44	4026.46	0.06
$\psi(4040)$	4039.00	2	2	58	116	4041.12	0.05	4041.12	0.05	4041.12	0.05	4061.47	0.56
$X(4050)^\pm$	4051.00	2	2	58	116	4041.12	0.24	4041.12	0.24	4041.12	0.24	4061.47	0.26
$X(4055)^\pm$	4054.00	2	2	58	116	4041.12	0.32	4041.12	0.32	4041.12	0.32	4061.47	0.18
$X(4100)^\pm$	4096.00	1	2	117	117	4078.95	0.42	4077.86	0.44	4075.96	0.49	4096.49	0.01

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\chi_{c1}(4140)$	4146.80	1	2	119	119	4148.62	0.04	4147.53	0.02	4145.63	0.03	4166.51	0.48
$\psi(4160)$	4191.00	2	2	60	120	4180.47	0.25	4180.47	0.25	4180.47	0.25	4201.52	0.25
X(4160)	4156.00	1	2	119	119	4148.62	0.18	4147.53	0.20	4145.63	0.25	4166.51	0.25
$Z_c(4200)$	4196.00	2	2	60	120	4180.47	0.37	4180.47	0.37	4180.47	0.37	4201.52	0.13
$\psi(4230)$	4220.00	1	2	121	121	4218.30	0.04	4217.21	0.07	4215.31	0.11	4236.54	0.39
$R_{c0}(4240)$	4239.00	2	2	61	122	4256.19	0.41	4253.94	0.35	4250.15	0.26	4271.55	0.77
$X(4250)^{\pm}$	4248.00	2	2	61	122	4256.19	0.19	4253.94	0.14	4250.15	0.05	4271.55	0.55
$\psi(4260)$	4220.00	1	2	121	121	4218.30	0.04	4217.21	0.07	4215.31	0.11	4236.54	0.39
$\chi_{c1}(4274)$	4274.00	2	2	61	122	4256.19	0.42	4253.94	0.47	4250.15	0.56	4271.55	0.06
X(4350)	4350.60	1	2	125	125	4357.65	0.16	4356.56	0.14	4354.66	0.09	4376.59	0.60
$\psi(4360)$	4368.00	1	2	125	125	4357.65	0.24	4356.56	0.26	4354.66	0.31	4376.59	0.20
$\psi(4390)$	4391.50	2	2	63	126	4395.54	0.09	4393.29	0.04	4389.50	0.05	4411.60	0.46
$\psi(4415)$	4421.00	1	2	127	127	4427.32	0.14	4426.23	0.12	4424.33	0.08	4446.61	0.58
$Z_c(4430)$	4478.00	2	2	64	128	4459.17	0.42	4459.17	0.42	4459.17	0.42	4481.63	0.08
$\chi_{c0}(4500)$	4506.00	1	2	129	129	4497.00	0.20	4495.91	0.22	4494.01	0.27	4516.64	0.24
$\psi(4660)$	4633.00	1	2	133	133	4636.35	0.07	4635.25	0.05	4633.36	0.01	4656.69	0.51
$\chi_{c0}(4700)$	4704.00	1	2	135	135	4706.02	0.04	4704.93	0.02	4703.03	0.02	4726.71	0.48
$b\bar{b}$:***													
$\eta_b(1S)$	9398.70	2	2	135	270	9412.10	0.14	9409.86	0.12	9406.06	0.08	9453.43	0.58
$\Upsilon(1S)$	9460.30	1	2	271	271	9443.90	0.17	9442.80	0.19	9440.90	0.21	9488.44	0.30
$\chi_{b0}(1P)$	9859.44	1	2	283	283	9861.95	0.03	9860.84	0.01	9858.95	0.01	9908.59	0.50
$\chi_{b1}(1P)$	9892.78	2	2	142	284	9893.78	0.01	9893.78	0.01	9893.78	0.01	9943.61	0.51
$h_b(1P)$	9899.30	2	2	142	284	9893.78	0.06	9893.78	0.06	9893.78	0.06	9943.61	0.45
$\chi_{b2}(1P)$	9912.21	2	2	142	284	9893.78	0.19	9893.78	0.19	9893.78	0.19	9943.61	0.32
$\eta_b(2S)$	9999.00	1	2	287	287	10,001.30	0.02	10,000.19	0.01	9998.30	0.01	10,048.64	0.50
$\Upsilon(2S)$	10,023.26	2	2	144	288	10,033.13	0.10	10,033.13	0.10	10,033.13	0.10	10,083.66	0.60
$\Upsilon_2(1D)$	10,163.70	2	2	146	292	10,172.48	0.09	10,172.48	0.09	10,172.48	0.09	10,223.71	0.59
$\chi_{b0}(2P)$	10,232.50	2	2	147	294	10,248.19	0.15	10,245.95	0.13	10,242.16	0.09	10,293.73	0.60
$\chi_{b1}(2P)$	10,255.46	2	2	147	294	10,248.19	0.07	10,245.95	0.09	10,242.16	0.13	10,293.73	0.37
$h_b(2P)$	10,259.80	2	2	147	294	10,248.19	0.11	10,245.95	0.13	10,242.16	0.17	10,293.73	0.33
$\chi_{b2}(2P)$	10,268.65	1	2	295	295	10,280.00	0.11	10,278.89	0.10	10,276.99	0.08	10,328.75	0.59
$\Upsilon(3S)$	10,355.20	1	2	297	297	10,349.67	0.05	10,348.57	0.06	10,346.67	0.08	10,398.77	0.42
$\chi_{b1}(3P)$	10,513.40	2	2	151	302	10,526.89	0.13	10,524.65	0.11	10,520.85	0.07	10,573.83	0.57
$\chi_{b2}(3P)$	10,524.00	2	2	151	302	10,526.89	0.03	10,524.65	0.01	10,520.85	0.03	10,573.83	0.47
$\Upsilon(4S)$	10,579.40	2	2	152	304	10,590.53	0.11	10,590.53	0.11	10,590.53	0.11	10,643.86	0.61
$Z_b(10610)$	10,607.20	2	2	152	304	10,590.53	0.16	10,590.53	0.16	10,590.53	0.16	10,643.86	0.35
$Z_b(10650)$	10,652.20	2	2	153	306	10,666.24	0.13	10,664.00	0.11	10,660.20	0.08	10,713.89	0.58
$\Upsilon(10753)$	10,752.70	1	2	309	309	10,767.72	0.14	10,766.61	0.13	10,764.72	0.11	10,818.92	0.62
$\Upsilon(10860)$	10,885.20	2	2	156	312	10,869.23	0.15	10,869.23	0.15	10,869.23	0.15	10,923.96	0.36

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
Υ(11020)	11,000.00	2	2	158	316	11,008.58	0.08	11,008.58	0.08	11,008.58	0.08	11,064.01	0.58
Baryons:													
p	938.27	1	2	27	27	943.50	0.56	942.51	0.45	940.61	0.25	945.34	0.75
n	939.57	1	2	27	27	943.50	0.42	942.51	0.31	940.61	0.11	945.34	0.61
N(1440)	1440.00	1	2	41	41	1431.26	0.61	1430.23	0.68	1428.33	0.81	1435.52	0.31
N(1520)	1515.00	1	2	43	43	1500.94	0.93	1499.90	1.00	1498.00	1.12	1505.55	0.62
N(1535)	1530.00	2	2	22	44	1532.84	0.19	1532.84	0.19	1532.84	0.19	1540.56	0.69
N(1650)	1650.00	1	2	47	47	1640.30	0.59	1639.25	0.65	1637.35	0.77	1645.60	0.27
N(1675)	1675.00	2	2	24	48	1672.19	0.17	1672.19	0.17	1672.19	0.17	1680.61	0.33
N(1680)	1685.00	2	2	24	48	1672.19	0.76	1672.19	0.76	1672.19	0.76	1680.61	0.26
N(1700)	1720.00	1	2	49	49	1709.97	0.58	1708.92	0.64	1707.03	0.75	1715.62	0.25
N(1710)	1710.00	1	2	49	49	1709.97	0.00	1708.92	0.06	1707.03	0.17	1715.62	0.33
N(1720)	1720.00	1	2	49	49	1709.97	0.58	1708.92	0.64	1707.03	0.75	1715.62	0.25
N(1875)	1875.00	2	2	27	54	1887.27	0.65	1885.01	0.53	1881.21	0.33	1890.69	0.84
N(1880)	1880.00	2	2	27	54	1887.27	0.39	1885.01	0.27	1881.21	0.06	1890.69	0.57
N(1895)	1895.00	2	2	27	54	1887.27	0.41	1885.01	0.53	1881.21	0.73	1890.69	0.23
N(1900)	1920.00	1	2	55	55	1919.00	0.05	1917.95	0.11	1916.05	0.21	1925.70	0.30
N(1990)	2020.00	2	2	29	58	2026.62	0.33	2024.36	0.22	2020.56	0.03	2030.74	0.53
N(2000)	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
N(2060)	2100.00	2	2	30	60	2090.24	0.46	2090.24	0.46	2090.24	0.46	2100.76	0.04
N(2100)	2100.00	2	2	30	60	2090.24	0.46	2090.24	0.46	2090.24	0.46	2100.76	0.04
N(2120)	2120.00	1	2	61	61	2128.03	0.38	2126.97	0.33	2125.07	0.24	2135.77	0.74
N(2190)	2180.00	2	2	31	62	2165.96	0.64	2163.71	0.75	2159.91	0.92	2170.79	0.42
N(2220)	2250.00	1	2	65	65	2267.39	0.77	2266.32	0.73	2264.42	0.64	2275.83	1.15
N(2250)	2280.00	1	2	65	65	2267.39	0.55	2266.32	0.60	2264.42	0.68	2275.83	0.18
N(2600)	2600.00	2	2	37	74	2584.01	0.62	2581.76	0.70	2577.96	0.85	2590.94	0.35
N(2700)	2612.00	1	2	75	75	2615.77	0.14	2614.69	0.10	2612.80	0.03	2625.95	0.53
Δ	1232.00	1	2	35	35	1222.22	0.79	1221.20	0.88	1219.30	1.03	1225.44	0.53
Δ(1600)	1570.00	1	2	45	45	1570.62	0.04	1569.58	0.03	1567.68	0.15	1575.57	0.35
Δ(1620)	1610.00	2	2	23	46	1608.57	0.09	1606.31	0.23	1602.51	0.46	1610.58	0.04
Δ(1700)	1710.00	1	2	49	49	1709.97	0.00	1708.92	0.06	1707.03	0.17	1715.62	0.33
Δ(1750)	n/a												
Δ(1900)	1860.00	1	2	53	53	1849.33	0.57	1848.27	0.63	1846.38	0.73	1855.67	0.23
Δ(1905)	1880.00	2	2	27	54	1887.27	0.39	1885.01	0.27	1881.21	0.06	1890.69	0.57
Δ(1910)	1900.00	2	2	27	54	1887.27	0.67	1885.01	0.79	1881.21	0.99	1890.69	0.49
Δ(1920)	1920.00	1	2	55	55	1919.00	0.05	1917.95	0.11	1916.05	0.21	1925.70	0.30
Δ(1930)	1950.00	2	2	28	56	1950.89	0.05	1950.89	0.05	1950.89	0.05	1960.71	0.55
Δ(1940)	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
Δ(1950)	1930.00	1	2	55	55	1919.00	0.57	1917.95	0.62	1916.05	0.72	1925.70	0.22

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\Delta(2000)$	2015.00	2	2	29	58	2026.62	0.58	2024.36	0.46	2020.56	0.28	2030.74	0.78
$\Delta(2150)$	2150.00	2	2	31	62	2165.96	0.74	2163.71	0.64	2159.91	0.46	2170.79	0.97
$\Delta(2200)$	2200.00	1	2	63	63	2197.71	0.10	2196.65	0.15	2194.75	0.24	2205.80	0.26
$\Delta(2300)$	2300.00	2	2	33	66	2305.31	0.23	2303.06	0.13	2299.26	0.03	2310.84	0.47
$\Delta(2350)$	2350.00	1	2	67	67	2337.06	0.55	2336.00	0.60	2334.10	0.68	2345.85	0.18
$\Delta(2390)$	2390.00	1	2	69	69	2406.74	0.70	2405.67	0.66	2403.77	0.58	2415.88	1.08
$\Delta(2400)$	2400.00	1	2	69	69	2406.74	0.28	2405.67	0.24	2403.77	0.16	2415.88	0.66
$\Delta(2420)$	2450.00	2	2	35	70	2444.66	0.22	2442.41	0.31	2438.61	0.46	2450.89	0.04
$\Delta(2750)$	2794.00	2	2	40	80	2786.98	0.25	2786.98	0.25	2786.98	0.25	2801.02	0.25
$\Delta(2950)$	2990.00	2	2	43	86	3002.05	0.40	2999.80	0.33	2996.01	0.20	3011.09	0.71
Λ	1115.68	2	2	16	32	1114.79	0.08	1114.79	0.08	1114.79	0.08	1120.41	0.42
$\Lambda(1405)$	1405.10	2	2	20	40	1393.49	0.83	1393.49	0.83	1393.49	0.83	1400.51	0.33
$\Lambda(1520)$	1519.00	2	2	22	44	1532.84	0.91	1532.84	0.91	1532.84	0.91	1540.56	1.42
$\Lambda(1600)$	1600.00	2	2	23	46	1608.57	0.54	1606.31	0.39	1602.51	0.16	1610.58	0.66
$\Lambda(1670)$	1674.00	2	2	24	48	1672.19	0.11	1672.19	0.11	1672.19	0.11	1680.61	0.39
$\Lambda(1690)$	1690.00	2	2	24	48	1672.19	1.05	1672.19	1.05	1672.19	1.05	1680.61	0.56
$\Lambda(1800)$	1800.00	2	2	26	52	1811.54	0.64	1811.54	0.64	1811.54	0.64	1820.66	1.15
$\Lambda(1810)$	1790.00	1	2	51	51	1779.65	0.58	1778.60	0.64	1776.70	0.74	1785.65	0.24
$\Lambda(1820)$	1820.00	2	2	26	52	1811.54	0.46	1811.54	0.46	1811.54	0.46	1820.66	0.04
$\Lambda(1830)$	1825.00	2	2	26	52	1811.54	0.74	1811.54	0.74	1811.54	0.74	1820.66	0.24
$\Lambda(1890)$	1890.00	2	2	27	54	1887.27	0.14	1885.01	0.26	1881.21	0.46	1890.69	0.04
$\Lambda(2000)$	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
$\Lambda(2050)$	2056.00	1	2	59	59	2058.36	0.11	2057.30	0.06	2055.40	0.03	2065.75	0.47
$\Lambda(2070)$	2070.00	1	2	59	59	2058.36	0.56	2057.30	0.61	2055.40	0.71	2065.75	0.21
$\Lambda(2080)$	2082.00	2	2	30	60	2090.24	0.40	2090.24	0.40	2090.24	0.40	2100.76	0.90
$\Lambda(2085)$	2020.00	2	2	29	58	2026.62	0.33	2024.36	0.22	2020.56	0.03	2030.74	0.53
$\Lambda(2100)$	2100.00	2	2	30	60	2090.24	0.46	2090.24	0.46	2090.24	0.46	2100.76	0.04
$\Lambda(2110)$	2090.00	2	2	30	60	2090.24	0.01	2090.24	0.01	2090.24	0.01	2100.76	0.51
$\Lambda(2325)$	2325.00	1	2	67	67	2337.06	0.52	2336.00	0.47	2334.10	0.39	2345.85	0.90
$\Lambda(2350)$	2350.00	1	2	67	67	2337.06	0.55	2336.00	0.60	2334.10	0.68	2345.85	0.18
$\Lambda(2585)$	2585.00	2	2	37	74	2584.01	0.04	2581.76	0.13	2577.96	0.27	2590.94	0.23
Σ^+	1189.38	2	2	17	34	1190.54	0.10	1188.27	0.09	1184.47	0.41	1190.43	0.09
Σ^0	1192.64	2	2	17	34	1190.54	0.18	1188.27	0.37	1184.47	0.69	1190.43	0.19
Σ^-	1197.45	2	2	17	34	1190.54	0.58	1188.27	0.77	1184.47	1.08	1190.43	0.59
$\Sigma(1385)^+$	1382.80	2	2	20	40	1393.49	0.77	1393.49	0.77	1393.49	0.77	1400.51	1.28
$\Sigma(1385)^0$	1383.70	2	2	20	40	1393.49	0.71	1393.49	0.71	1393.49	0.71	1400.51	1.21
$\Sigma(1385)^-$	1387.20	2	2	20	40	1393.49	0.45	1393.49	0.45	1393.49	0.45	1400.51	0.96
$\Sigma(1660)$	1660.00	2	2	24	48	1672.19	0.73	1672.19	0.73	1672.19	0.73	1680.61	1.24
$\Sigma(1670)$	1675.00	2	2	24	48	1672.19	0.17	1672.19	0.17	1672.19	0.17	1680.61	0.33

Table 2: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\Sigma(1750)$	1750.00	2	2	25	50	1747.92	0.12	1745.66	0.25	1741.86	0.46	1750.63	0.04
$\Sigma(1775)$	1775.00	1	2	51	51	1779.65	0.26	1778.60	0.20	1776.70	0.10	1785.65	0.60
$\Sigma(1910)$	1910.00	1	2	55	55	1919.00	0.47	1917.95	0.42	1916.05	0.32	1925.70	0.82
$\Sigma(1915)$	1915.00	1	2	55	55	1919.00	0.21	1917.95	0.15	1916.05	0.05	1925.70	0.56
$\Sigma(2030)$	2030.00	2	2	29	58	2026.62	0.17	2024.36	0.28	2020.56	0.46	2030.74	0.04
$\Sigma(2250)$	2250.00	1	2	65	65	2267.39	0.77	2266.32	0.73	2264.42	0.64	2275.83	1.15
Ξ^0	1314.86	2	2	19	38	1329.88	1.14	1327.62	0.97	1323.82	0.68	1330.48	1.19
Ξ^-	1321.72	2	2	19	38	1329.88	0.62	1327.62	0.45	1323.82	0.16	1330.48	0.66
$\Xi(1530)^0$	1531.80	2	2	22	44	1532.84	0.07	1532.84	0.07	1532.84	0.07	1540.56	0.57
$\Xi(1530)^-$	1535.00	2	2	22	44	1532.84	0.14	1532.84	0.14	1532.84	0.14	1540.56	0.36
$\Xi(1690)$	1690.00	2	2	24	48	1672.19	1.05	1672.19	1.05	1672.19	1.05	1680.61	0.56
$\Xi(1820)$	1823.00	2	2	26	52	1811.54	0.63	1811.54	0.63	1811.54	0.63	1820.66	0.13
$\Xi(1950)$	1950.00	2	2	28	56	1950.89	0.05	1950.89	0.05	1950.89	0.05	1960.71	0.55
$\Xi(2030)$	2025.00	2	2	29	58	2026.62	0.08	2024.36	0.03	2020.56	0.22	2030.74	0.28
$\Omega(1672)^-$	1672.45	2	2	24	48	1672.19	0.02	1672.19	0.02	1672.19	0.02	1680.61	0.49
$\Omega(2012)^-$	2012.40	2	2	29	58	2026.62	0.71	2024.36	0.59	2020.56	0.41	2030.74	0.91
$\Omega(2250)^-$	2252.00	1	2	65	65	2267.39	0.68	2266.32	0.64	2264.42	0.55	2275.83	1.06
charmed, bottom:													
Λ_c^+	2286.46	1	2	65	65	2267.39	0.83	2266.32	0.88	2264.42	0.96	2275.83	0.47
$\Sigma_c(2455)$	2453.75	2	2	35	70	2444.66	0.37	2442.41	0.46	2438.61	0.62	2450.89	0.12
Ξ_c^0	2470.91	1	2	71	71	2476.41	0.22	2475.34	0.18	2473.45	0.10	2485.90	0.61
Ω_c^0	2695.20	1	2	77	77	2685.44	0.36	2684.37	0.40	2682.47	0.47	2695.98	0.03
Λ_b^0	5619.60	1	2	161	161	5611.80	0.14	5610.70	0.16	5608.80	0.19	5637.04	0.31
Σ_b	5813.10	1	2	167	167	5820.82	0.13	5819.72	0.11	5817.82	0.08	5847.12	0.59
Ξ_b^0	5791.90	2	2	83	166	5789.03	0.05	5786.78	0.09	5782.99	0.15	5812.11	0.35
Ω_b^-	6046.10	1	2	173	173	6029.84	0.27	6028.75	0.29	6026.85	0.32	6057.20	0.18
Heavy Bosons:													
m_{gb}	42891.7#	2	4	9	18	43,591.2	1.63	43,270.9	0.88	42,750.4	0.33	43,182.0	0.68
W	80,379.0	2	4	17	34	81,582.8	1.50	81,269.8	1.11	80,750.7	0.46	81,566.0	1.48
Z	91,187.6	2	4	19	38	91,081.9	0.12	90,769.7	0.46	90,250.8	1.03	91,162.0	0.03
H^0	125,100.0	1	4	53	53	126,269.6	0.93	126,135.0	0.83	125,876.1	0.62	127,147.0	1.64
Average error Δ:							0.43		0.43		0.44		0.53
Ratio with 'statistical' error Δ/Δ_s:							0.88		0.87		0.89		1.08

Column 2 contains the empirical masses, E_1 and E_2 are calculated according to Eq. (70), E_3 according to Eq. (71), all three with parameters $f_{ext} = -2.1573$, $s = 1.089$. E_4 according to (Eq. (71) et sqq. with $\exp(\pi a) := 1/2\alpha = 68.518$, i.e. an approximation which corresponds to the phenomenological approach of Section 2. The error columns display the percentaged discrepancies between calculation and empirical mass. *: For even terms ($N = 2n$) always $C = 0$ holds (no contribution of $F_{(3)}$). **: Masses in the constituent quark model. ***: + possibly non $q\bar{q}$ states. #: 0.5 $(m_W + m_Z)/2 \approx$ Mac Gregor's gauge boson mass unit $m_{gb} = m_{u,d}/\alpha$.

Table 3: Calculation of rest energies (masses) in MeV for scenario 2: Only even numbers of mass units ($m = 2$) for bosons and only odd numbers of mass units ($m = 1$) for fermions allowed.

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
Leptons:													
e	0.511	1	0	1	1	0.511	0.00	0.511	0.00	0.511	0.00	0.511	0.00
μ	105.66	1	2	3	3	106.26	0.57	106.43	0.73	104.51	1.09	105.04	0.59
τ	1776.86	1	2	51	51	1779.65	0.16	1778.60	0.10	1776.70	0.01	1785.65	0.49
Quarks:													
u, d	315.00**	1	2	9	9	316.16	0.37	315.44	0.14	313.54	0.46	315.11	0.04
s	525.00**	1	2	15	15	525.34	0.07	524.46	0.10	522.56	0.46	525.19	0.04
c	1270.00	1	2	37	37	1291.90	1.72	1290.88	1.64	1288.98	1.49	1295.47	2.01
b	4180.00	1	2	119	119	4148.62	0.75	4147.53	0.78	4145.63	0.82	4166.51	0.32
t	172,760.0	1	4	73	73	173,774.9	0.59	173,635.4	0.51	173,376.5	0.36	175,127.0	1.37
Mesons:													
unflavored:													
$\pi^\pm$	139.57	2	2	2	4	139.35	0.16	139.35	0.16	139.35	0.16	140.05	0.34
π^0	134.98	2	2	2	4	139.35	3.24	139.35	3.24	139.35	3.24	140.05	3.76
η	547.86	2	2	8	16	557.40	1.74	557.40	1.74	557.40	1.74	560.20	2.25
$f_0(500)$	400–550	2	2	7	14	493.84		491.53		487.72		490.18	
$\rho(770)$	775.26	2	2	11	22	772.51	0.36	770.22	0.65	766.42	1.14	770.28	0.64
$\omega(782)$	782.65	2	2	11	22	772.51	1.30	770.22	1.59	766.42	2.07	770.28	1.58
$\eta'(958)$	957.78	2	2	14	28	975.44	1.84	975.44	1.84	975.44	1.84	980.36	2.36
$f_0(980)$	990.00	2	2	14	28	975.44	1.47	975.44	1.47	975.44	1.47	980.36	0.97
$a_0(980)$	980.00	2	2	14	28	975.44	0.46	975.44	0.46	975.44	0.46	980.36	0.04
$\phi(1020)$	1019.46	2	2	15	30	1051.19	3.11	1048.92	2.89	1045.12	2.52	1050.38	3.03
$h_1(1170)$	1166.00	2	2	17	34	1190.54	2.10	1188.27	1.91	1184.47	1.58	1190.43	2.10
$b_1(1235)$	1229.50	2	2	18	36	1254.14	2.00	1254.14	2.00	1254.14	2.00	1260.46	2.52
$a_1(1260)$	1230.00	2	2	18	36	1254.14	1.96	1254.14	1.96	1254.14	1.96	1260.46	2.48
$f_2(1270)$	1275.50	2	2	18	36	1254.14	1.67	1254.14	1.67	1254.14	1.67	1260.46	1.18
$f_1(1285)$	1281.90	2	2	18	36	1254.14	2.17	1254.14	2.17	1254.14	2.17	1260.46	1.67
$\eta(1295)$	1294.00	2	2	19	38	1329.88	2.77	1327.62	2.60	1323.82	2.30	1330.48	2.82
$\pi(1300)$	1300.00	2	2	19	38	1329.88	2.30	1327.62	2.12	1323.82	1.83	1330.48	2.34
$a_2(1320)$	1316.90	2	2	19	38	1329.88	0.99	1327.62	0.81	1323.82	0.53	1330.48	1.03
$f_0(1370)$	1200–1500	2	2	19	38	1329.88		1327.62		1323.82		1330.48	
$\pi_1(1400)$	1354.00	2	2	19	38	1329.88	1.78	1327.62	1.95	1323.82	2.23	1330.48	1.74
$\eta(1405)$	1408.80	2	2	20	40	1393.49	1.09	1393.49	1.09	1393.49	1.09	1400.51	0.59
$h_1(1415)$	1416.00	2	2	20	40	1393.49	1.59	1393.49	1.59	1393.49	1.59	1400.51	1.09
$a_1(1420)$	1411.00	2	2	20	40	1393.49	1.24	1393.49	1.24	1393.49	1.24	1400.51	0.74
$f_1(1420)$	1426.30	2	2	20	40	1393.49	2.30	1393.49	2.30	1393.49	2.30	1400.51	1.81

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\omega(1420)$	1410.00	2	2	20	40	1393.49	1.17	1393.49	1.17	1393.49	1.17	1400.51	0.67
$f_2(1430)$	1430.00	2	2	20	40	1393.49	2.55	1393.49	2.55	1393.49	2.55	1400.51	2.06
$a_0(1450)$	1474.00	2	2	21	42	1469.23	0.32	1466.96	0.48	1463.17	0.74	1470.53	0.24
$\rho(1450)$	1465.00	2	2	21	42	1469.23	0.29	1466.96	0.13	1463.17	0.13	1470.53	0.38
$\eta(1475)$	1475.00	2	2	21	42	1469.23	0.39	1466.96	0.54	1463.17	0.80	1470.53	0.30
$f_0(1500)$	1506.00	2	2	22	44	1532.84	1.78	1532.84	1.78	1532.84	1.78	1540.56	2.29
$f_1(1510)$	1518.00	2	2	22	44	1532.84	0.98	1532.84	0.98	1532.84	0.98	1540.56	1.49
$f_2'(1525)$	1517.40	2	2	22	44	1532.84	1.02	1532.84	1.02	1532.84	1.02	1540.56	1.53
$f_2(1565)$	1542.00	2	2	22	44	1532.84	0.59	1532.84	0.59	1532.84	0.59	1540.56	0.09
$\rho(1570)$	1570.00	2	2	22	44	1532.84	2.37	1532.84	2.37	1532.84	2.37	1540.56	1.88
$h_1(1595)$	1594.00	2	2	23	46	1608.57	0.91	1606.31	0.77	1602.51	0.53	1610.58	1.04
$\pi_1(1600)$	1660.00	2	2	24	48	1672.19	0.73	1672.19	0.73	1672.19	0.73	1680.61	1.24
$a_1(1640)$	1655.00	2	2	24	48	1672.19	1.04	1672.19	1.04	1672.19	1.04	1680.61	1.55
$f_2(1640)$	1639.00	2	2	23	46	1608.57	1.86	1606.31	1.99	1602.51	2.23	1610.58	1.73
$\eta_2(1645)$	1617.00	2	2	23	46	1608.57	0.52	1606.31	0.66	1602.51	0.90	1610.58	0.40
$\omega(1650)$	1670.00	2	2	24	48	1672.19	0.13	1672.19	0.13	1672.19	0.13	1680.61	0.64
$\omega_3(1670)$	1667.00	2	2	24	48	1672.19	0.31	1672.19	0.31	1672.19	0.31	1680.61	0.82
$\pi_2(1670)$	1670.60	2	2	24	48	1672.19	0.10	1672.19	0.10	1672.19	0.10	1680.61	0.60
$\phi(1680)$	1680.00	2	2	24	48	1672.19	0.46	1672.19	0.46	1672.19	0.46	1680.61	0.04
$\rho_3(1690)$	1688.80	2	2	24	48	1672.19	0.98	1672.19	0.98	1672.19	0.98	1680.61	0.48
$\rho(1700)$	1720.00	2	2	25	50	1747.92	1.62	1745.66	1.49	1741.86	1.27	1750.63	1.78
$a_2(1700)$	1705.00	2	2	24	48	1672.19	1.92	1672.19	1.92	1672.19	1.92	1680.61	1.43
$f_0(1710)$	1704.00	2	2	24	48	1672.19	1.87	1672.19	1.87	1672.19	1.87	1680.61	1.37
$\eta(1760)$	1751.00	2	2	25	50	1747.92	0.18	1745.66	0.30	1741.86	0.52	1750.63	0.02
$\pi(1800)$	1810.00	2	2	26	52	1811.54	0.08	1811.54	0.08	1811.54	0.08	1820.66	0.59
$f_2(1810)$	1815.00	2	2	26	52	1811.54	0.19	1811.54	0.19	1811.54	0.19	1820.66	0.31
X(1835)	1826.50	2	2	26	52	1811.54	0.82	1811.54	0.82	1811.54	0.82	1820.66	0.32
$\phi_3(1850)$	1854.00	2	2	27	54	1887.27	1.79	1885.01	1.67	1881.21	1.47	1890.69	1.98
$\eta_2(1870)$	1842.00	2	2	26	52	1811.54	1.65	1811.54	1.65	1811.54	1.65	1820.66	1.16
$\pi_2(1880)$	1874.00	2	2	27	54	1887.27	0.71	1885.01	0.59	1881.21	0.38	1890.69	0.89
$\rho(1900)$	1900.00	2	2	27	54	1887.27	0.67	1885.01	0.79	1881.21	0.99	1890.69	0.49
$f_2(1910)$	1900.00	2	2	27	54	1887.27	0.67	1885.01	0.79	1881.21	0.99	1890.69	0.49
$a_0(1950)$	1931.00	2	2	28	56	1950.89	1.03	1950.89	1.03	1950.89	1.03	1960.71	1.54
$f_2(1950)$	1936.00	2	2	28	56	1950.89	0.77	1950.89	0.77	1950.89	0.77	1960.71	1.28
$a_4(1970)$	1967.00	2	2	28	56	1950.89	0.82	1950.89	0.82	1950.89	0.82	1960.71	0.32
$\rho_3(1990)$	1990.00	2	2	29	58	2026.62	1.84	2024.36	1.73	2020.56	1.54	2030.74	2.05
$\pi_2(2005)$	1963.00	2	2	28	56	1950.89	0.62	1950.89	0.62	1950.89	0.62	1960.71	0.12
$f_2(2010)$	2011.00	2	2	29	58	2026.62	0.78	2024.36	0.66	2020.56	0.48	2030.74	0.98
$f_0(2020)$	1992.00	2	2	29	58	2026.62	1.74	2024.36	1.62	2020.56	1.43	2030.74	1.94
$f_4(2050)$	2018.00	2	2	29	58	2026.62	0.43	2024.36	0.32	2020.56	0.13	2030.74	0.63

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\pi_2(2100)$	2090.00	2	2	30	60	2090.24	0.01	2090.24	0.01	2090.24	0.01	2100.76	0.51
$f_0(2100)$	2086.00	2	2	30	60	2090.24	0.20	2090.24	0.20	2090.24	0.20	2100.76	0.71
$f_2(2150)$	2157.00	2	2	31	62	2165.96	0.42	2163.71	0.31	2159.91	0.13	2170.79	0.64
$\rho(2150)$	2150.00	2	2	31	62	2165.96	0.74	2163.71	0.64	2159.91	0.46	2170.79	0.97
$\phi(2170)$	2160.00	2	2	31	62	2165.96	0.28	2163.71	0.17	2159.91	0.00	2170.79	0.50
$f_0(2200)$	2187.00	2	2	31	62	2165.96	0.96	2163.71	1.06	2159.91	1.24	2170.79	0.74
$f_J(2220)$	2231.10	2	2	32	64	2229.59	0.07	2229.59	0.07	2229.59	0.07	2240.81	0.44
$\eta(2225)$	2221.00	2	2	32	64	2229.59	0.39	2229.59	0.39	2229.59	0.39	2240.81	0.89
$\rho_3(2250)$	2250.00	2	2	32	64	2229.59	0.91	2229.59	0.91	2229.59	0.91	2240.81	0.41
$f_2(2300)$	2297.00	2	2	33	66	2305.31	0.36	2303.06	0.26	2299.26	0.10	2310.84	0.60
$f_4(2300)$	2300.00	2	2	33	66	2305.31	0.23	2303.06	0.13	2299.26	0.03	2310.84	0.47
$f_0(2330)$	2330.00	2	2	33	66	2305.31	1.06	2303.06	1.16	2299.26	1.32	2310.84	0.82
$f_2(2340)$	2345.00	2	2	34	68	2368.93	1.02	2368.93	1.02	2368.93	1.02	2380.86	1.53
$\rho_5(2350)$	2330.00	2	2	33	66	2305.31	1.06	2303.06	1.16	2299.26	1.32	2310.84	0.82
$f_6(2510)$	2465.00	2	2	35	70	2444.66	0.83	2442.41	0.92	2438.61	1.07	2450.89	0.57
strange:													
$K^\pm$	493.68	2	2	7	14	493.84	0.03	491.53	0.43	487.72	1.21	490.18	0.71
K^0	497.61	2	2	7	14	493.84	0.76	491.53	1.22	487.72	1.99	490.18	1.49
$K_0^\star(700)$	630–730	2	2	10	20	696.75		696.75		696.75		700.25	
$K^\star(892)$	894.24	2	2	13	26	911.85	1.97	909.57	1.71	905.77	1.29	910.33	1.80
$K_1(1270)$	1253.00	2	2	18	36	1254.14	0.09	1254.14	0.09	1254.14	0.09	1260.46	0.60
$K_1(1400)$	1403.00	2	2	20	40	1393.49	0.68	1393.49	0.68	1393.49	0.68	1400.51	0.18
$K^\star(1410)$	1414.00	2	2	20	40	1393.49	1.45	1393.49	1.45	1393.49	1.45	1400.51	0.95
$K_0^\star(1430)$	1425.00	2	2	20	40	1393.49	2.21	1393.49	2.21	1393.49	2.21	1400.51	1.72
$K_2^\star(1430)$	1429.85	2	2	20	40	1393.49	2.54	1393.49	2.54	1393.49	2.54	1400.51	2.05
K(1460)	1460.00	2	2	21	42	1469.23	0.63	1466.96	0.48	1463.17	0.22	1470.53	0.72
$K_2(1580)$	1580.00	2	2	23	46	1608.57	1.81	1606.31	1.67	1602.51	1.42	1610.58	1.94
K(1630)	1629.00	2	2	23	46	1608.57	1.25	1606.31	1.39	1602.51	1.63	1610.58	1.13
$K_1(1650)$	1672.00	2	2	24	48	1672.19	0.01	1672.19	0.01	1672.19	0.01	1680.61	0.51
$K^\star(1680)$	1718.00	2	2	25	50	1747.92	1.74	1745.66	1.61	1741.86	1.39	1750.63	1.90
$K_2(1770)$	1773.00	2	2	25	50	1747.92	1.41	1745.66	1.54	1741.86	1.76	1750.63	1.26
$K_3^\star(1780)$	1776.00	2	2	25	50	1747.92	1.58	1745.66	1.71	1741.86	1.92	1750.63	1.43
$K_2(1820)$	1819.00	2	2	26	52	1811.54	0.41	1811.54	0.41	1811.54	0.41	1820.66	0.09
K(1830)	1874.00	2	2	27	54	1887.27	0.71	1885.01	0.59	1881.21	0.38	1890.69	0.89
$K_0^\star(1950)$	1945.00	2	2	28	56	1950.89	0.30	1950.89	0.30	1950.89	0.30	1960.71	0.81
$K_2^\star(1980)$	1943.00	2	2	28	56	1950.89	0.41	1950.89	0.41	1950.89	0.41	1960.71	0.91
$K_4^\star(2045)$	2048.00	2	2	29	58	2026.62	1.04	2024.36	1.15	2020.56	1.34	2030.74	0.84
$K_2(2250)$	2247.00	2	2	32	64	2229.59	0.78	2229.59	0.78	2229.59	0.78	2240.81	0.28
$K_3(2320)$	2324.00	2	2	33	66	2305.31	0.80	2303.06	0.90	2299.26	1.06	2310.84	0.57

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$K_5^\star(2380)$	2382.00	2	2	34	68	2368.93	0.55	2368.93	0.55	2368.93	0.55	2380.86	0.05
$K_4(2500)$	2490.00	2	2	36	72	2508.28	0.73	2508.28	0.73	2508.28	0.73	2520.91	1.24
K(3100)	3100.00	2	2	44	88	3065.68	1.11	3065.68	1.11	3065.68	1.11	3081.12	0.61
charmed:													
$D^\pm$	1869.65	2	2	27	54	1887.27	0.94	1885.01	0.82	1881.21	0.62	1890.69	1.13
D^0	1864.83	2	2	27	54	1887.27	1.20	1885.01	1.08	1881.21	0.88	1890.69	1.39
$D^\star(2007)^0$	2006.85	2	2	29	58	2026.62	0.98	2024.36	0.87	2020.56	0.68	2030.74	1.19
$D^\star(2010)^\pm$	2010.26	2	2	29	58	2026.62	0.81	2024.36	0.70	2020.56	0.51	2030.74	1.02
$D_0^\star(2300)^0$	2300.00	2	2	33	66	2305.31	0.23	2303.06	0.13	2299.26	0.03	2310.84	0.47
$D_0^\star(2300)^\pm$	2349.00	2	2	34	68	2368.93	0.85	2368.93	0.85	2368.93	0.85	2380.86	1.36
$D_1(2420)^0$	2420.50	2	2	35	70	2444.66	1.00	2442.41	0.91	2438.61	0.75	2450.89	1.26
$D_1(2420)^\pm$	2423.20	2	2	35	70	2444.66	0.89	2442.41	0.79	2438.61	0.64	2450.89	1.14
$D_1(2430)^0$	2427.00	2	2	35	70	2444.66	0.73	2442.41	0.63	2438.61	0.48	2450.89	0.98
$D_2^\star(2460)^0$	2460.56	2	2	35	70	2444.66	0.65	2442.41	0.74	2438.61	0.89	2450.89	0.39
$D_2^\star(2460)^\pm$	2465.40	2	2	35	70	2444.66	0.84	2442.41	0.93	2438.61	1.09	2450.89	0.59
$D(2550)^0$	2564.00	2	2	37	74	2584.01	0.78	2581.76	0.69	2577.96	0.54	2590.94	1.05
$D_J^\star(2600)$	2623.00	2	2	38	76	2647.63	0.94	2647.63	0.94	2647.63	0.94	2660.97	1.45
$D^\star(2640)^\pm$	2637.00	2	2	38	76	2647.63	0.40	2647.63	0.40	2647.63	0.40	2660.97	0.91
$D(2740)^0$	2737.00	2	2	39	78	2723.36	0.50	2721.10	0.58	2717.31	0.72	2730.99	0.22
$D_3^\star(2750)$	2763.50	2	2	40	80	2786.98	0.85	2786.98	0.85	2786.98	0.85	2801.02	1.36
$D(3000)^0$	3214.00	2	2	46	92	3205.03	0.28	3205.03	0.28	3205.03	0.28	3221.17	0.22
charmed, strange:													
$D_s^\pm$	1968.34	2	2	28	56	1950.89	0.89	1950.89	0.89	1950.89	0.89	1960.71	0.39
$D_s^{\star\pm}$	2112.20	2	2	30	60	2090.24	1.04	2090.24	1.04	2090.24	1.04	2100.76	0.54
$D_{s0}^\star(2317)^\pm$	2317.80	2	2	33	66	2305.31	0.54	2303.06	0.64	2299.26	0.80	2310.84	0.30
$D_{s1}(2460)^\pm$	2459.50	2	2	35	70	2444.66	0.60	2442.41	0.69	2438.61	0.85	2450.89	0.35
$D_{s1}(2536)^\pm$	2535.11	2	2	36	72	2508.28	1.06	2508.28	1.06	2508.28	1.06	2520.91	0.56
$D_{s2}^\star(2573)$	2569.10	2	2	37	74	2584.01	0.58	2581.76	0.49	2577.96	0.34	2590.94	0.85
$D_{s1}^\star(2700)^\pm$	2708.30	2	2	39	78	2723.36	0.56	2721.10	0.47	2717.31	0.33	2730.99	0.84
$D_{s1}^\star(2860)^\pm$	2859.00	2	2	41	82	2862.70	0.13	2860.45	0.05	2856.66	0.08	2871.04	0.42
$D_{s3}^\star(2860)^\pm$	2860.50	2	2	41	82	2862.70	0.08	2860.45	0.00	2856.66	0.13	2871.04	0.37
$D_{sJ}(3040)^\pm$	3044.00	2	2	44	88	3065.68	0.71	3065.68	0.71	3065.68	0.71	3081.12	1.22
bottom:													
$B^\pm$	5279.34	2	2	76	152	5295.26	0.30	5295.26	0.30	5295.26	0.30	5321.93	0.81
B^0	5279.65	2	2	76	152	5295.26	0.30	5295.26	0.30	5295.26	0.30	5321.93	0.80
$B^\star$	5324.70	2	2	76	152	5295.26	0.55	5295.26	0.55	5295.26	0.55	5321.93	0.05
$B_1(5721)^+$	5725.90	2	2	82	164	5713.31	0.22	5713.31	0.22	5713.31	0.22	5742.08	0.28
$B_1(5721)^0$	5726.10	2	2	82	164	5713.31	0.22	5713.31	0.22	5713.31	0.22	5742.08	0.28

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$B_J^\star(5732)$	5698.00	2	2	82	164	5713.31	0.27	5713.31	0.27	5713.31	0.27	5742.08	0.77
$B_2^\star(5747)^+$	5737.20	2	2	82	164	5713.31	0.42	5713.31	0.42	5713.31	0.42	5742.08	0.09
$B_2^\star(5747)^0$	5739.50	2	2	82	164	5713.31	0.46	5713.31	0.46	5713.31	0.46	5742.08	0.04
$B_J(5840)^+$	5851.00	2	2	84	168	5852.66	0.03	5852.66	0.03	5852.66	0.03	5882.13	0.53
$B_J(5840)^0$	5863.00	2	2	84	168	5852.66	0.18	5852.66	0.18	5852.66	0.18	5882.13	0.33
$B_J(5970)^+$	5964.00	2	2	86	172	5992.01	0.47	5992.01	0.47	5992.01	0.47	6022.18	0.98
$B_J(5970)^0$	5971.00	2	2	86	172	5992.01	0.35	5992.01	0.35	5992.01	0.35	6022.18	0.86
bottom, strange:													
B_s^0	5366.88	2	2	77	154	5370.98	0.08	5368.74	0.03	5364.94	0.04	5391.96	0.47
$B_s^\star$	5415.40	2	2	78	156	5434.61	0.35	5434.61	0.35	5434.61	0.35	5461.98	0.86
$X(5568)^\pm$	5566.90	2	2	80	160	5573.96	0.13	5573.96	0.13	5573.96	0.13	5602.03	0.63
$B_{s1}(5830)^0$	5828.70	2	2	84	168	5852.66	0.41	5852.66	0.41	5852.66	0.41	5882.13	0.92
$B_{s2}^\star(5840)^0$	5839.86	2	2	84	168	5852.66	0.22	5852.66	0.22	5852.66	0.22	5882.13	0.72
$B_{sJ}^\star(5850)$	5853.00	2	2	84	168	5852.66	0.01	5852.66	0.01	5852.66	0.01	5882.13	0.50
bottom, charmed:													
B_c^+	6274.90	2	2	90	180	6270.71	0.07	6270.71	0.07	6270.71	0.07	6302.29	0.44
$B_c(2S)^\pm$	6871.60	2	2	99	198	6903.82	0.47	6901.57	0.44	6897.78	0.38	6932.51	0.89
$c\bar{c}$:***													
$\eta_c(1S)$	2983.90	2	2	43	86	3002.05	0.61	2999.80	0.53	2996.01	0.41	3011.09	0.91
$J/\psi(1S)$	3096.90	2	2	44	88	3065.68	1.01	3065.68	1.01	3065.68	1.01	3081.12	0.51
$\chi_{c0}(1P)$	3414.71	2	2	49	98	3420.10	0.16	3417.85	0.09	3414.05	0.02	3431.24	0.48
$\chi_{c1}(1P)$	3510.67	2	2	50	100	3483.73	0.77	3483.73	0.77	3483.73	0.77	3501.27	0.27
$h_c(1P)$	3525.38	2	2	51	102	3559.45	0.97	3557.20	0.90	3553.40	0.79	3571.30	1.30
$\chi_{c2}(1P)$	3556.17	2	2	51	102	3559.45	0.09	3557.20	0.03	3553.40	0.08	3571.30	0.43
$\eta_c(2S)$	3637.50	2	2	52	104	3623.08	0.40	3623.08	0.40	3623.08	0.40	3641.32	0.11
$\psi(2S)$	3686.10	2	2	53	106	3698.79	0.34	3696.55	0.28	3692.75	0.18	3711.35	0.68
$\psi(3770)$	3773.70	2	2	54	108	3762.42	0.30	3762.42	0.30	3762.42	0.30	3781.37	0.20
$\psi_2(3823)$	3822.20	2	2	55	110	3838.14	0.42	3835.90	0.36	3832.10	0.26	3851.40	0.76
$\psi_3(3842)$	3842.71	2	2	55	110	3838.14	0.12	3835.90	0.18	3832.10	0.28	3851.40	0.23
$\chi_{c0}(3860)$	3862.00	2	2	55	110	3838.14	0.62	3835.90	0.68	3832.10	0.77	3851.40	0.27
$\chi_{c1}(3872)$	3871.69	2	2	56	112	3901.77	0.78	3901.77	0.78	3901.77	0.78	3921.42	1.28
$Z_c(3900)$	3888.40	2	2	56	112	3901.77	0.34	3901.77	0.34	3901.77	0.34	3921.42	0.85
X(3915)	3918.40	2	2	56	112	3901.77	0.42	3901.77	0.42	3901.77	0.42	3921.42	0.08
$\chi_{c2}(3930)$	3922.20	2	2	56	112	3901.77	0.52	3901.77	0.52	3901.77	0.52	3921.42	0.02
X(3940)	3942.00	2	2	57	114	3977.49	0.90	3975.25	0.84	3971.45	0.75	3991.45	1.25
$X(4020)^\pm$	4024.10	2	2	58	116	4041.12	0.42	4041.12	0.42	4041.12	0.42	4061.47	0.93
$\psi(4040)$	4039.00	2	2	58	116	4041.12	0.05	4041.12	0.05	4041.12	0.05	4061.47	0.56
$X(4050)^\pm$	4051.00	2	2	58	116	4041.12	0.24	4041.12	0.24	4041.12	0.24	4061.47	0.26
$X(4055)^\pm$	4054.00	2	2	58	116	4041.12	0.32	4041.12	0.32	4041.12	0.32	4061.47	0.18

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$X(4100)^{\pm}$	4096.00	2	2	59	118	4116.84	0.51	4114.59	0.45	4110.80	0.36	4131.50	0.87
$\chi_{c1}(4140)$	4146.80	2	2	59	118	4116.84	0.72	4114.59	0.78	4110.80	0.87	4131.50	0.37
$\psi(4160)$	4191.00	2	2	60	120	4180.47	0.25	4180.47	0.25	4180.47	0.25	4201.52	0.25
X(4160)	4156.00	2	2	60	120	4180.47	0.59	4180.47	0.59	4180.47	0.59	4201.52	1.10
$Z_c(4200)$	4196.00	2	2	60	120	4180.47	0.37	4180.47	0.37	4180.47	0.37	4201.52	0.13
$\psi(4230)$	4220.00	2	2	61	122	4256.19	0.86	4253.94	0.80	4250.15	0.71	4271.55	1.22
$R_{c0}(4240)$	4239.00	2	2	61	122	4256.19	0.41	4253.94	0.35	4250.15	0.26	4271.55	0.77
$X(4250)^{\pm}$	4248.00	2	2	61	122	4256.19	0.19	4253.94	0.14	4250.15	0.05	4271.55	0.55
$\psi(4260)$	4220.00	2	2	61	122	4256.19	0.86	4253.94	0.80	4250.15	0.71	4271.55	1.22
$\chi_{c1}(4274)$	4274.00	2	2	61	122	4256.19	0.42	4253.94	0.47	4250.15	0.56	4271.55	0.06
X(4350)	4350.60	2	2	62	124	4319.82	0.71	4319.82	0.71	4319.82	0.71	4341.57	0.21
$\psi(4360)$	4368.00	2	2	63	126	4395.54	0.63	4393.29	0.58	4389.50	0.49	4411.60	1.00
$\psi(4390)$	4391.50	2	2	63	126	4395.54	0.09	4393.29	0.04	4389.50	0.05	4411.60	0.46
$\psi(4415)$	4421.00	2	2	63	126	4395.54	0.58	4393.29	0.63	4389.50	0.71	4411.60	0.21
$Z_c(4430)$	4478.00	2	2	64	128	4459.17	0.42	4459.17	0.42	4459.17	0.42	4481.63	0.08
$\chi_{c0}(4500)$	4506.00	2	2	65	130	4534.89	0.64	4532.64	0.59	4528.84	0.51	4551.65	1.01
$\psi(4660)$	4633.00	2	2	66	132	4598.52	0.74	4598.52	0.74	4598.52	0.74	4621.68	0.24
$\chi_{c0}(4700)$	4704.00	2	2	67	134	4674.24	0.63	4671.99	0.68	4668.19	0.76	4691.70	0.26
$b\bar{b}$:***													
$\eta_b(1S)$	9398.70	2	2	135	270	9412.10	0.14	9409.86	0.12	9406.06	0.08	9453.43	0.58
$\Upsilon(1S)$	9460.30	2	2	136	272	9475.74	0.16	9475.74	0.16	9475.74	0.16	9523.45	0.67
$\chi_{b0}(1P)$	9859.44	2	2	141	282	9830.15	0.30	9827.91	0.32	9824.11	0.36	9873.58	0.14
$\chi_{b1}(1P)$	9892.78	2	2	142	284	9893.78	0.01	9893.78	0.01	9893.78	0.01	9943.61	0.51
$h_b(1P)$	9899.30	2	2	142	284	9893.78	0.06	9893.78	0.06	9893.78	0.06	9943.61	0.45
$\chi_{b2}(1P)$	9912.21	2	2	142	284	9893.78	0.19	9893.78	0.19	9893.78	0.19	9943.61	0.32
$\eta_b(2S)$	9999.00	2	2	144	288	10,033.13	0.34	10,033.13	0.34	10,033.13	0.34	10,083.66	0.85
$\Upsilon(2S)$	10,023.26	2	2	144	288	10,033.13	0.10	10,033.13	0.10	10,033.13	0.10	10,083.66	0.60
$\Upsilon_2(1D)$	10,163.70	2	2	146	292	10,172.48	0.09	10,172.48	0.09	10,172.48	0.09	10,223.71	0.59
$\chi_{b0}(2P)$	10,232.50	2	2	147	294	10,248.19	0.15	10,245.95	0.13	10,242.16	0.09	10,293.73	0.60
$\chi_{b1}(2P)$	10,255.46	2	2	147	294	10,248.19	0.07	10,245.95	0.09	10,242.16	0.13	10,293.73	0.37
$h_b(2P)$	10,259.80	2	2	147	294	10,248.19	0.11	10,245.95	0.13	10,242.16	0.17	10,293.73	0.33
$\chi_{b2}(2P)$	10,268.65	2	2	147	294	10,248.19	0.20	10,245.95	0.22	10,242.16	0.26	10,293.73	0.24
$\Upsilon(3S)$	10,355.20	2	2	149	298	10,387.54	0.31	10,385.30	0.29	10,381.51	0.25	10,433.78	0.76
$\chi_{b1}(3P)$	10,513.40	2	2	151	302	10,526.89	0.13	10,524.65	0.11	10,520.85	0.07	10,573.83	0.57
$\chi_{b2}(3P)$	10,524.00	2	2	151	302	10,526.89	0.03	10,524.65	0.01	10,520.85	0.03	10,573.83	0.47
$\Upsilon(4S)$	10,579.40	2	2	152	304	10,590.53	0.11	10,590.53	0.11	10,590.53	0.11	10,643.86	0.61
$Z_b(10610)$	10,607.20	2	2	152	304	10,590.53	0.16	10,590.53	0.16	10,590.53	0.16	10,643.86	0.35
$Z_b(10650)$	10,652.20	2	2	153	306	10,666.24	0.13	10,664.00	0.11	10,660.20	0.08	10,713.89	0.58
$\Upsilon(10753)$	10,752.70	2	2	154	308	10,729.88	0.21	10,729.88	0.21	10,729.88	0.21	10,783.91	0.29
$\Upsilon(10860)$	10,885.20	2	2	156	312	10,869.23	0.15	10,869.23	0.15	10,869.23	0.15	10,923.96	0.36

Table 3: (continued)

Particle	Mass (MeV)	*m*	$\underline{n}$	*N*	*mN*	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
ϒ(11020)	11,000.00	2	2	158	316	11,008.58	0.08	11,008.58	0.08	11,008.58	0.08	11,064.01	0.58
Baryons:													
p	938.27	1	2	27	27	943.50	0.56	942.51	0.45	940.61	0.25	945.34	0.75
n	939.57	1	2	27	27	943.50	0.42	942.51	0.31	940.61	0.11	945.34	0.61
N(1440)	1440.00	1	2	41	41	1431.26	0.61	1430.23	0.68	1428.33	0.81	1435.52	0.31
N(1520)	1515.00	1	2	43	43	1500.94	0.93	1499.90	1.00	1498.00	1.12	1505.55	0.62
N(1535)	1530.00	1	2	43	43	1500.94	1.90	1499.90	1.97	1498.00	2.09	1505.55	1.60
N(1650)	1650.00	1	2	47	47	1640.30	0.59	1639.25	0.65	1637.35	0.77	1645.60	0.27
N(1675)	1675.00	1	2	47	47	1640.30	2.07	1639.25	2.13	1637.35	2.25	1645.60	1.76
N(1680)	1685.00	1	2	49	49	1709.97	1.48	1708.92	1.42	1707.03	1.31	1715.62	1.82
N(1700)	1720.00	1	2	49	49	1709.97	0.58	1708.92	0.64	1707.03	0.75	1715.62	0.25
N(1710)	1710.00	1	2	49	49	1709.97	0.00	1708.92	0.06	1707.03	0.17	1715.62	0.33
N(1720)	1720.00	1	2	49	49	1709.97	0.58	1708.92	0.64	1707.03	0.75	1715.62	0.25
N(1875)	1875.00	1	2	53	53	1849.33	1.37	1848.27	1.43	1846.38	1.53	1855.67	1.03
N(1880)	1880.00	1	2	53	53	1849.33	1.63	1848.27	1.69	1846.38	1.79	1855.67	1.29
N(1895)	1895.00	1	2	55	55	1919.00	1.27	1917.95	1.21	1916.05	1.11	1925.70	1.62
N(1900)	1920.00	1	2	55	55	1919.00	0.05	1917.95	0.11	1916.05	0.21	1925.70	0.30
N(1990)	2020.00	1	2	57	57	1988.68	1.55	1987.62	1.60	1985.72	1.70	1995.72	1.20
N(2000)	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
N(2060)	2100.00	1	2	61	61	2128.03	1.33	2126.97	1.28	2125.07	1.19	2135.77	1.70
N(2100)	2100.00	1	2	61	61	2128.03	1.33	2126.97	1.28	2125.07	1.19	2135.77	1.70
N(2120)	2120.00	1	2	61	61	2128.03	0.38	2126.97	0.33	2125.07	0.24	2135.77	0.74
N(2190)	2180.00	1	2	63	63	2197.71	0.81	2196.65	0.76	2194.75	0.68	2205.80	1.18
N(2220)	2250.00	1	2	65	65	2267.39	0.77	2266.32	0.73	2264.42	0.64	2275.83	1.15
N(2250)	2280.00	1	2	65	65	2267.39	0.55	2266.32	0.60	2264.42	0.68	2275.83	0.18
N(2600)	2600.00	1	2	75	75	2615.77	0.61	2614.69	0.57	2612.80	0.49	2625.95	1.00
N(2700)	2612.00	1	2	75	75	2615.77	0.14	2614.69	0.10	2612.80	0.03	2625.95	0.53
Δ	1232.00	1	2	35	35	1222.22	0.79	1221.20	0.88	1219.30	1.03	1225.44	0.53
Δ(1600)	1570.00	1	2	45	45	1570.62	0.04	1569.58	0.03	1567.68	0.15	1575.57	0.35
Δ(1620)	1610.00	1	2	47	47	1640.30	1.88	1639.25	1.82	1637.35	1.70	1645.60	2.21
Δ(1700)	1710.00	1	2	49	49	1709.97	0.00	1708.92	0.06	1707.03	0.17	1715.62	0.33
Δ(1750)	n/a												
Δ(1900)	1860.00	1	2	53	53	1849.33	0.57	1848.27	0.63	1846.38	0.73	1855.67	0.23
Δ(1905)	1880.00	1	2	53	53	1849.33	1.63	1848.27	1.69	1846.38	1.79	1855.67	1.29
Δ(1910)	1900.00	1	2	55	55	1919.00	1.00	1917.95	0.94	1916.05	0.84	1925.70	1.35
Δ(1920)	1920.00	1	2	55	55	1919.00	0.05	1917.95	0.11	1916.05	0.21	1925.70	0.30
Δ(1930)	1950.00	1	2	55	55	1919.00	1.59	1917.95	1.64	1916.05	1.74	1925.70	1.25
Δ(1940)	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
Δ(1950)	1930.00	1	2	55	55	1919.00	0.57	1917.95	0.62	1916.05	0.72	1925.70	0.22

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\Delta(2000)$	2015.00	1	2	57	57	1988.68	1.31	1987.62	1.36	1985.72	1.45	1995.72	0.96
$\Delta(2150)$	2150.00	1	2	61	61	2128.03	1.02	2126.97	1.07	2125.07	1.16	2135.77	0.66
$\Delta(2200)$	2200.00	1	2	63	63	2197.71	0.10	2196.65	0.15	2194.75	0.24	2205.80	0.26
$\Delta(2300)$	2300.00	1	2	65	65	2267.39	1.42	2266.32	1.46	2264.42	1.55	2275.83	1.05
$\Delta(2350)$	2350.00	1	2	67	67	2337.06	0.55	2336.00	0.60	2334.10	0.68	2345.85	0.18
$\Delta(2390)$	2390.00	1	2	69	69	2406.74	0.70	2405.67	0.66	2403.77	0.58	2415.88	1.08
$\Delta(2400)$	2400.00	1	2	69	69	2406.74	0.28	2405.67	0.24	2403.77	0.16	2415.88	0.66
$\Delta(2420)$	2450.00	1	2	71	71	2476.41	1.08	2475.34	1.03	2473.45	0.96	2485.90	1.47
$\Delta(2750)$	2794.00	1	2	81	81	2824.79	1.10	2823.72	1.06	2821.82	1.00	2836.03	1.50
$\Delta(2950)$	2990.00	1	2	85	85	2964.14	0.86	2963.07	0.90	2961.17	0.96	2976.08	0.47
Λ	1115.68	1	2	31	31	1082.86	2.94	1081.85	3.03	1079.96	3.20	1085.39	2.71
$\Lambda(1405)$	1405.10	1	2	41	41	1431.26	1.86	1430.23	1.79	1428.33	1.65	1435.52	2.17
$\Lambda(1520)$	1519.00	1	2	43	43	1500.94	1.19	1499.90	1.26	1498.00	1.38	1505.55	0.89
$\Lambda(1600)$	1600.00	1	2	45	45	1570.62	1.84	1569.58	1.90	1567.68	2.02	1575.57	1.53
$\Lambda(1670)$	1674.00	1	2	47	47	1640.30	2.01	1639.25	2.08	1637.35	2.19	1645.60	1.70
$\Lambda(1690)$	1690.00	1	2	49	49	1709.97	1.18	1708.92	1.12	1707.03	1.01	1715.62	1.52
$\Lambda(1800)$	1800.00	1	2	51	51	1779.65	1.13	1778.60	1.19	1776.70	1.29	1785.65	0.80
$\Lambda(1810)$	1790.00	1	2	51	51	1779.65	0.58	1778.60	0.64	1776.70	0.74	1785.65	0.24
$\Lambda(1820)$	1820.00	1	2	53	53	1849.33	1.61	1848.27	1.55	1846.38	1.45	1855.67	1.96
$\Lambda(1830)$	1825.00	1	2	53	53	1849.33	1.33	1848.27	1.28	1846.38	1.17	1855.67	1.68
$\Lambda(1890)$	1890.00	1	2	55	55	1919.00	1.53	1917.95	1.48	1916.05	1.38	1925.70	1.89
$\Lambda(2000)$	2000.00	1	2	57	57	1988.68	0.57	1987.62	0.62	1985.72	0.71	1995.72	0.21
$\Lambda(2050)$	2056.00	1	2	59	59	2058.36	0.11	2057.30	0.06	2055.40	0.03	2065.75	0.47
$\Lambda(2070)$	2070.00	1	2	59	59	2058.36	0.56	2057.30	0.61	2055.40	0.71	2065.75	0.21
$\Lambda(2080)$	2082.00	1	2	59	59	2058.36	1.14	2057.30	1.19	2055.40	1.28	2065.75	0.78
$\Lambda(2085)$	2020.00	1	2	57	57	1988.68	1.55	1987.62	1.60	1985.72	1.70	1995.72	1.20
$\Lambda(2100)$	2100.00	1	2	61	61	2128.03	1.33	2126.97	1.28	2125.07	1.19	2135.77	1.70
$\Lambda(2110)$	2090.00	1	2	59	59	2058.36	1.51	2057.30	1.56	2055.40	1.66	2065.75	1.16
$\Lambda(2325)$	2325.00	1	2	67	67	2337.06	0.52	2336.00	0.47	2334.10	0.39	2345.85	0.90
$\Lambda(2350)$	2350.00	1	2	67	67	2337.06	0.55	2336.00	0.60	2334.10	0.68	2345.85	0.18
$\Lambda(2585)$	2585.00	1	2	75	75	2615.77	1.19	2614.69	1.15	2612.80	1.08	2625.95	1.58
Σ^+	1189.38	1	2	35	35	1222.22	2.76	1221.20	2.68	1219.30	2.52	1225.44	3.03
Σ^0	1192.64	1	2	35	35	1222.22	2.48	1221.20	2.39	1219.30	2.24	1225.44	2.75
Σ^-	1197.45	1	2	35	35	1222.22	2.07	1221.20	1.98	1219.30	1.83	1225.44	2.34
$\Sigma(1385)^+$	1382.80	1	2	39	39	1361.58	1.53	1360.55	1.61	1358.65	1.75	1365.50	1.25
$\Sigma(1385)^0$	1383.70	1	2	39	39	1361.58	1.60	1360.55	1.67	1358.65	1.81	1365.50	1.32
$\Sigma(1385)^-$	1387.20	1	2	39	39	1361.58	1.85	1360.55	1.92	1358.65	2.06	1365.50	1.56
$\Sigma(1660)$	1660.00	1	2	47	47	1640.30	1.19	1639.25	1.25	1637.35	1.36	1645.60	0.87
$\Sigma(1670)$	1675.00	1	2	47	47	1640.30	2.07	1639.25	2.13	1637.35	2.25	1645.60	1.76

Table 3: (continued)

Particle	Mass (MeV)	m	$\underline{n}$	N	mN	E_1 $C = 0.9^*$	Error %	E_2 $C = 1.0^*$	Error %	E_3	Error %	E_4	Error %
$\Sigma(1750)$	1750.00	1	2	51	51	1779.65	1.69	1778.60	1.63	1776.70	1.53	1785.65	2.04
$\Sigma(1775)$	1775.00	1	2	51	51	1779.65	0.26	1778.60	0.20	1776.70	0.10	1785.65	0.60
$\Sigma(1910)$	1910.00	1	2	55	55	1919.00	0.47	1917.95	0.42	1916.05	0.32	1925.70	0.82
$\Sigma(1915)$	1915.00	1	2	55	55	1919.00	0.21	1917.95	0.15	1916.05	0.05	1925.70	0.56
$\Sigma(2030)$	2030.00	1	2	59	59	2058.36	1.40	2057.30	1.34	2055.40	1.25	2065.75	1.76
$\Sigma(2250)$	2250.00	1	2	65	65	2267.39	0.77	2266.32	0.73	2264.42	0.64	2275.83	1.15
Ξ^0	1314.86	1	2	37	37	1291.90	1.75	1290.88	1.82	1288.98	1.97	1295.47	1.47
Ξ^-	1321.72	1	2	37	37	1291.90	2.26	1290.88	2.33	1288.98	2.48	1295.47	1.99
$\Xi(1530)^0$	1531.80	1	2	43	43	1500.94	2.01	1499.90	2.08	1498.00	2.21	1505.55	1.71
$\Xi(1530)^-$	1535.00	1	2	43	43	1500.94	2.22	1499.90	2.29	1498.00	2.41	1505.55	1.92
$\Xi(1690)$	1690.00	1	2	49	49	1709.97	1.18	1708.92	1.12	1707.03	1.01	1715.62	1.52
$\Xi(1820)$	1823.00	1	2	53	53	1849.33	1.44	1848.27	1.39	1846.38	1.28	1855.67	1.79
$\Xi(1950)$	1950.00	1	2	55	55	1919.00	1.59	1917.95	1.64	1916.05	1.74	1925.70	1.25
$\Xi(2030)$	2025.00	1	2	59	59	2058.36	1.65	2057.30	1.59	2055.40	1.50	2065.75	2.01
$\Omega(1672)^-$	1672.45	1	2	47	47	1640.30	1.92	1639.25	1.99	1637.35	2.10	1645.60	1.61
$\Omega(2012)^-$	2012.40	1	2	57	57	1988.68	1.18	1987.62	1.23	1985.72	1.33	1995.72	0.83
$\Omega(2250)^-$	2252.00	1	2	65	65	2267.39	0.68	2266.32	0.64	2264.42	0.55	2275.83	1.06
charmed, bottom:													
Λ_c^+	2286.46	1	2	65	65	2267.39	0.83	2266.32	0.88	2264.42	0.96	2275.83	0.47
$\Sigma_c(2455)$	2453.75	1	2	71	71	2476.41	0.92	2475.34	0.88	2473.45	0.80	2485.90	1.31
Ξ_c^0	2470.91	1	2	71	71	2476.41	0.22	2475.34	0.18	2473.45	0.10	2485.90	0.61
Ω_c^0	2695.20	1	2	77	77	2685.44	0.36	2684.37	0.40	2682.47	0.47	2695.98	0.03
Λ_b^0	5619.60	1	2	161	161	5611.80	0.14	5610.70	0.16	5608.80	0.19	5637.04	0.31
Σ_b	5813.10	1	2	167	167	5820.82	0.13	5819.72	0.11	5817.82	0.08	5847.12	0.59
Ξ_b^0	5791.90	1	2	167	167	5820.82	0.50	5819.72	0.48	5817.82	0.45	5847.12	0.95
Ω_b^-	6046.10	1	2	173	173	6029.84	0.27	6028.75	0.29	6026.85	0.32	6057.20	0.18
Heavy Bosons:													
m_{gb}	42891.7#	2	4	9	18	43,591.2	1.63	43,270.9	0.88	42,750.4	0.33	43,182.0	0.68
W	80,379.0	2	4	17	34	81,582.8	1.50	81,269.8	1.11	80,750.7	0.46	81,566.0	1.48
Z	91,187.6	2	4	19	38	91,081.9	0.12	90,769.7	0.46	90,250.8	1.03	91,162.0	0.03
H^0	125,100.0	2	4	26	52	123,501.0	1.28	123,501.0	1.28	123,501.0	1.28	124,748.0	0.28
Average error Δ:							0.85		0.84		0.85		0.91
Ratio with 'statistical' error Δ/Δ_s:							0.86		0.85		0.86		0.92

Column 2 contains the empirical masses, E_1 and E_2 are calculated according to Eq. (70), E_3 according to Eq. (71), all three with parameters $f_{ext} = -2.1573$, $s = 1.089$. E_4 according to Eq. (71) et sqq. with $\exp(\pi a) := 1/2\alpha = 68.518$, i.e. an approximation which corresponds to the phenomenological approach of Section 2. The error columns display the percentaged discrepancies between calculation and empirical mass. * For even terms ($N = 2n$) always $C = 0$ holds (no contribution of $F_{(3)}$). **: Masses in the constituent quark model. ***: + possibly non $q\bar{q}$ states. #: $0.5\,(m_W + m_Z)/2 \approx$ Mac Gregor's gauge boson mass unit $m_{gb} = m_{u,d}/\alpha$.

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